Invariant measure for the stochastic Ginzburg Landau equation

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Abstract. The existence of martingale solutions and stationary solutions of stochastic Ginzburg-Landau equations under general hypothesizes on the dimension, the non linear term and the added noise is investigated. With a few more assumptions, it is established that the transition semi-group is well defined and that the stationary martingale solution yields the existence of an invariant measure. Moreover this invariant measure is shown to be unique.

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1 Introduction

This paper is about the stochastic Ginzburg Landau equation with Dirichlet boundary conditions on a bounded domain $D$ in $\mathbb{R}^d$:

$$du - (1 + i\nu)\Delta_d u \, dt - \alpha \cdot u \, dt + (1 + i\mu)|u|^{2\sigma} u \, dt = \Phi dW,$$

with $\alpha \geq 0$, and $\Phi$ an operator from $L^2(D)$ to $L^2(D)$ satisfying some assumptions that will be specified further. This equation can be found in many domains of physics like wave propagation or fluid mechanics, it is also mathematically very interesting as a model of parabolic equation. Besides, the Schrödinger equation, according to the coefficients, can be seen as a limit of Ginzburg-Landau equations, see for instance [1]. The stochastic Ginzburg-Landau equation can also be used as a model equation, simpler than the Navier Stokes equations, in the study of turbulence phenomena, see the work of S.B. Kuksin [17].
The fact that in physical experiments there are always small irregularities which give birth to a new random phenomenon, justifies the study of equations with noise. The process $\Phi(W(t + dt) - W(t))$ is gaussian, with values in $L^2(D)$, its quadratic variation is $\Phi^* \Phi dt$ and its correlation is formally:

$$E \left[ \frac{d\Phi W}{dt}(x, t) \frac{d\Phi W}{dt}(y, s) \right] = C_\Phi(x, y) \delta(t - s).$$

When $C_\Phi(x, y) = \delta(x - y)$ the noise is said to be white in space and time (space time white noise), but it is not always possible to treat such a rough noise. The smoother $\Phi$ is, the more regular the correlation $C_\Phi$ is; but it is more interesting to assume weak regularity on the correlation to be closer to the space time white noise. This might be more relevant in physical applications [21].

Note that also for Schrödinger equation a space-time white noise should be taken into account. However, due to the absence of viscous term it is totally impossible to treat this equation with non smooth $\Phi$. On the contrary, the Ginzburg-Landau equation is parabolic and we are able to consider a space-time white noise in space dimension one and slightly correlated noises in higher space dimension.

We introduce $Z$, solution of the linear equation:

$$dZ - (1 + i\nu)\Delta_d dZ dt = \Phi dW.$$ 

Under our assumptions, $Z$ has paths with continuous variations with respect to $(x, t)$. Hence introducing the translated unknown $v = u - Z$, we are led to the equation:

$$\frac{dv}{dt}(t) = (1 + i\nu)(\Delta_d + \alpha I)v - (1 + i\mu)|v + Z|^{2\sigma}(v + Z) + \alpha Z,$$

which can be solved with deterministic methods. It is not difficult to see that, for $d = 1$ and any $\sigma$ or for $d = 2$ and small $\sigma$, a fixed point method can be used to construct a global solution in $L^2(D)$. But in general this method fails and it is only possible to get, thanks to compactness methods, a solution which might not be unique. Compactness methods in the stochastic setting use Krylov-Bogoliubov and Skohorod theorems and lead to martingale solutions, that is to say “weak” solutions in the probabilistic sense, see [20]. Here we follow the approach of F. Flandoli and D. Gatarek in [13] to construct a martingale solution for general $d$ and $\sigma$ for the stochastic Ginzburg-Landau equation.

Moreover we wish to investigate the asymptotic behavior. Uniqueness in law is not proved and we are not able to define a transition semi-group. Therefore, in the spirit of F. Flandoli and D. Gatarek [13], instead of establishing the existence of an invariant measure, we construct a stationary solution in $L^2(D)$. Nevertheless, under some restrictions on $\nu$, it is possible to prove in the space $L^p(D)$, the uniqueness of the solution of the stochastic Ginzburg-Landau equation, using the ideas of C.R. Doering, J.D. Gibbon and C.D. Levermore in [9] in the deterministic case. Then it is natural to wonder if the uniqueness of a global $L^p$ solution for