AFFINE LINES ON $\mathbb{Q}$-HOMOLOGY PLANES
AND GROUP ACTIONS

MIKHAIL ZAIDENBERG

Université Grenoble I
Institut Fourier
UMR 5582 du CNRS, 38402
St.-Martin d’Hères Cédex, France
zaidenbe@ujf-grenoble.fr

Abstract. This paper is a supplement to the papers [KiKo] and [GMMR]. We show the role of group actions in the classification of affine lines on $\mathbb{Q}$-homology planes.

Introduction

This paper is a supplement to the papers [KiKo] and [GMMR]. Our aim is to shed a light on the role of group actions in the classification of affine lines on $\mathbb{Q}$-homology planes with logarithmic Kodaira dimension $-\infty$. This enables us to strengthen certain results in loc. sit. (see Section 1).

Let us fix terminology. It is usual [Mi, Chap. 3, §4] to call a smooth $\mathbb{Q}$-acyclic ($\mathbb{Z}$-acyclic, respectively) surface over $\mathbb{C}$ a $\mathbb{Q}$-homology plane (a homology plane, respectively). By Fujita’s lemma [Fu, 2.5] such a surface is necessarily affine. Likewise we call a homology line an irreducible affine curve $\Gamma$ with Euler characteristic $\epsilon(\Gamma) = 1$. So $\Gamma$ is homeomorphic to $\mathbb{R}^2$ and its normalization is isomorphic to $\mathbb{A}^1 = \mathbb{A}^1_{\mathbb{C}}$. A smooth normal curve isomorphic to $\mathbb{A}^1$ will be called an affine line. Following [Mi] we let $\mathbb{A}^1_* = \mathbb{A}^1 \setminus \{0\}$. As usual $k$ stands for logarithmic Kodaira dimension.

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1. Main results

Theorem 1. Let $X$ be a $\mathbb{Q}$-homology plane and $\Gamma$ a homology line on $X$. Then the following hold:

(a) Suppose that $k(X \setminus \Gamma) = -\infty$. Then $\Gamma$ is either an orbit of an effective $\mathbb{C}_+$-action on $X$ or a connected component of the fixed point set of such an action. Anyhow $\Gamma \simeq \mathbb{A}^1$ is a fiber component of the corresponding orbit map (an $\mathbb{A}^1$-ruling) $\pi : X \to \mathbb{A}^1$.

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(b) Suppose that \( \bar{k}(X \setminus \Gamma) \geq 0 \). Suppose further that \( \Gamma \simeq \mathbb{A}^1 \) and \( \bar{k}(X) = -\infty \). Then \( \Gamma \) is an orbit closure of an effective hyperbolic \( \mathbb{C}^* \)-action on \( X \). Moreover, \( X \) admits an effective action of a semidirect product \( G = \mathbb{C}^* \rtimes \mathbb{C}_+ \) with an open orbit \( U \). The orbit map \( X \to \mathbb{A}^1 \) of the induced \( \mathbb{A}^1 \)-action defines an \( \mathbb{A}^1 \)-ruling on \( X \) with a unique multiple fiber, say \( \Gamma' \simeq \mathbb{A}^1 \), such that \( \Gamma \) and \( \Gamma' \) meet at one point transversally and \( U = X \setminus \Gamma' \simeq \mathbb{A}^1 \times \mathbb{A}^1 \). Furthermore, this \( \mathbb{A}^1 \)-action moves \( \Gamma \). Consequently, there exists a continuous family of affine lines \( \Gamma_t \) on \( X \) with the same properties as \( \Gamma \).

(c) Suppose that \( \Gamma \) is singular. Then \( X \simeq \mathbb{A}^2 \) and \( \bar{k}(X \setminus \Gamma) = 1 \). Moreover,\(^1\) there is an isomorphism \( X \simeq \mathbb{A}^2 \) sending \( \Gamma \) to a curve \( V(x^k - y^l) \) with coprime \( k, l \geq 2 \). Consequently, \( \Gamma \) is an orbit closure of an elliptic \( \mathbb{C}^* \)-action on \( X \).

We indicate below a proof of the theorem. The cases (a), (b), and (c) are proven in Sections 2, 3, and 4, respectively. Besides, in cases (a) and (b) we provide in Lemmas 3, and 7, respectively, a description of the pairs \((X, \Gamma)\) satisfying their assumptions. The assertion of (b) follows from Theorem 1.1 in [KiKo], cf. also Theorem 3.10 in [GMMR]. In the case of a \( \mathbb{Z} \)-homology plane (c) was established in [Za]; the proof for a \( \mathbb{Q} \)-homology plane is similar. This gives a strengthening of Theorem 1.3 in [KiKo].

Cases (a)–(c) of Theorem 1 do not exhaust all the possibilities for the pair \((X, \Gamma)\) as above. To complete the picture let us summarize some known facts, see, e.g., [Za], [GuPa], [Mi, Chap. 3, §4], and the references therein.

**Theorem 2.** We let, as before, \( X \) be a \( \mathbb{Q} \)-homology plane and \( \Gamma \subseteq X \) a homology line. If \( \Gamma \) is singular then \((X, \Gamma)\) is as in Theorem 1(c). Suppose further that \( \Gamma \) is smooth, i.e., is an affine line. Then \( \bar{k}(X) \leq \bar{k}(X \setminus \Gamma) \leq 1^2 \) and one of the following cases occurs:

- (a) \((\bar{k}(X), \bar{k}(X \setminus \Gamma)) = (-\infty, -\infty) \) and \((X, \Gamma)\) is as in Theorem 1(a), that is, \( \Gamma \) is of fiber type and \( X \setminus \Gamma \) carries a family of disjoint affine lines.\(^3\)
- (b) \((\bar{k}(X), \bar{k}(X \setminus \Gamma)) = (-\infty, 0) \) or \((-\infty, 1) \) and \((X, \Gamma)\) is as in Theorem 1(b).\(^4\)
- (c) \((\bar{k}(X), \bar{k}(X \setminus \Gamma)) = (0, 0) \) and either \( X \) is not NC minimal or \( X \) is one of the Fujita’s surfaces \( H[-k, k] \) \((k \geq 1)\).\(^5\) Anyhow, \( \Gamma \) is a unique affine line on \( X \) unless \( X = H[-1, 1] \).
- (d) \((\bar{k}(X), \bar{k}(X \setminus \Gamma)) = (0, 1) \), \( X = H[-1, 1] \) and there are exactly two affine lines, say, \( \Gamma_0 \) and \( \Gamma = \Gamma_1 \) on \( X \). These lines meet transversally in two distinct points, moreover, \( \bar{k}(X \setminus \Gamma_0) = 0 \) and \( \bar{k}(X \setminus \Gamma_1) = 1 \).
- (e) \((\bar{k}(X), \bar{k}(X \setminus \Gamma)) = (1, 1) \), there is a unique \( \mathbb{A}^2 \)-fibration on \( X \), and \( \Gamma \) is a fiber component of its degenerate fiber.\(^6\) There can be at most one further affine line on \( X \), which is then another component of this same degenerate fiber, and these components meet transversally at one point.

**Remark 1.** Let \( X \) be a \( \mathbb{Z} \)-homology plane. By [Fu] then \( \bar{k}(X) \neq 0 \). By [Za, supplement] \( \bar{k}(X) = 1 \) if and only if there exists a unique homology (in fact, affine) line on \( X \).

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\(^1\)**This** due to the Lin–Zaidenberg theorem [LiZa], [Mi, Chap. 3, §3].

\(^2\)See [Mi, Chap.2, Theorem 6.7.1].

\(^3\)See also Lemma 3 below.

\(^4\)The both possibilities actually occur, see the Construction in Section 3 and also Lemma 7.

\(^5\)We refer, e.g., to [Fu], [GuPa], [Mi, Chaps. 3, 4.4.1–4.4.2] for definitions.

\(^6\)The same conclusions hold also in case (c) if \( X \) is not NC-minimal [GuPa].