YANGIANS AND MICKELSSON ALGEBRAS I

SERGEY KHOROSHKIN
Institute for Theoretical and Experimental Physics
Moscow 117259, Russia
khor@itep.ru

MAXIM NAZAROV
Department of Mathematics
University of York
York YO10 5DD, England
mln1@york.ac.uk

Abstract. We study the composition of the functor from the category of modules over the Lie algebra \( \mathfrak{gl}_m \) to the category of modules over the degenerate affine Hecke algebra of \( \text{GL}_N \) introduced by I. Cherednik, with the functor from the latter category to the category of modules over the Yangian \( Y(\mathfrak{gl}_n) \) due to V. Drinfeld. We propose a representation theoretic explanation of a link between the intertwining operators on the tensor products of \( Y(\mathfrak{gl}_n) \)-modules, and the “extremal cocycle” on the Weyl group of \( \mathfrak{gl}_m \) defined by D. Zhelobenko. We also establish a connection between the composition of the functors, and the “centralizer construction” of the Yangian \( Y(\mathfrak{gl}_n) \) discovered by G. Olshanski.

0. Introduction

The central role in this article is played by two well-known constructions. One of these constructions is due to V. Drinfeld [D2]. Let \( H_N \) be the degenerate affine Hecke algebra corresponding to the general linear group \( \text{GL}_N \) over a non-Archimedean local field. This is an associative algebra over the complex field \( \mathbb{C} \) which contains the symmetric group ring \( \mathbb{C}S_N \) as a subalgebra. Let \( Y(\mathfrak{gl}_n) \) be the Yangian of the general linear Lie algebra \( \mathfrak{gl}_n \). This Yangian is a Hopf algebra over the field \( \mathbb{C} \) which contains the universal enveloping algebra \( U(\mathfrak{gl}_n) \) as a subalgebra. In [D2] for any \( H_N \)-module \( W \), an action of the algebra \( Y(\mathfrak{gl}_n) \) was defined on the vector space \( (W \otimes (\mathbb{C}^m)^{\otimes N})^{S_N} \) of the diagonal \( S_N \)-invariants in the tensor product of the vector spaces \( W \) and \( (\mathbb{C}^n)^{\otimes N} \). Details of this construction are reproduced in Section 1 of the present article.

The other construction that we use here is due to I. Cherednik [C2], it was also studied by T. Arakawa, T. Suzuki and A. Tsuchiya [AST]. For any module \( V \) over the Lie algebra \( \mathfrak{gl}_m \), it provides an action of the algebra \( H_N \) on the tensor product of \( \mathfrak{gl}_m \)-modules \( V \otimes (\mathbb{C}^m)^{\otimes N} \). This action of \( H_N \) commutes with the diagonal action of \( \mathfrak{gl}_m \) on the tensor product. Details of this construction are also reproduced in Section 1 of the present article. By applying the construction from [D2] to the \( H_N \)-module \( W = V \otimes (\mathbb{C}^m)^{\otimes N} \), we get an action of the Yangian \( Y(\mathfrak{gl}_m) \) on

\[
(V \otimes (\mathbb{C}^m)^{\otimes N} \otimes (\mathbb{C}^n)^{\otimes N})^{S_N} = V \otimes \mathbb{S}^N(\mathbb{C}^m \otimes \mathbb{C}^n)
\]

commuting with the action of the Lie algebra \( \mathfrak{gl}_m \); see our Proposition 1.3. By taking the direct sum over \( N = 0, 1, 2, \ldots \) of these \( Y(\mathfrak{gl}_n) \)-modules, we turn into an \( Y(\mathfrak{gl}_n) \)-module.
the vector space $V \otimes S(\mathbb{C}^m \otimes \mathbb{C}^n)$. It is also a $\mathfrak{gl}_m$-module; we denote this bimodule by $\mathcal{E}_m(V)$. The additive group $\mathbb{C}$ acts on the Hopf algebra $Y(\mathfrak{gl}_n)$ by automorphisms. We denote by $\mathcal{E}_m(V)$ the $Y(\mathfrak{gl}_n)$-module obtained from $\mathcal{E}_m(V)$ via pull-back through the automorphism of $Y(\mathfrak{gl}_n)$ corresponding to the element $z \in \mathbb{C}$. As a $\mathfrak{gl}_m$-module $\mathcal{E}_m(V)$ coincides with $\mathcal{E}_m(V)$. In this article, we identify the symmetric algebra $S(\mathbb{C}^m \otimes \mathbb{C}^n)$ with the ring $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$ of polynomial functions on the vector space $\mathbb{C}^m \otimes \mathbb{C}^n$.

Now take the Lie algebra $\mathfrak{gl}_{m+1}$. Let $\mathfrak{p}$ be the maximal parabolic subalgebra of $\mathfrak{gl}_{m+1}$ containing the direct sum of Lie algebras $\mathfrak{gl}_m \oplus \mathfrak{gl}_1$. Let $\mathfrak{q}$ be the Abelian subalgebra of $\mathfrak{gl}_{m+1}$ such that $\mathfrak{gl}_{m+1} = \mathfrak{q} \oplus \mathfrak{p}$. For any $\mathfrak{gl}_1$-module $U$ let $V \boxtimes U$ be the $\mathfrak{gl}_{m+1}$-module parabolically induced from the $\mathfrak{gl}_m \oplus \mathfrak{gl}_1$-module $\mathcal{E}_m(V \boxtimes U)$. This is a module induced from the subalgebra $\mathfrak{p}$. Consider the space $\mathcal{E}_{m+1}(V \boxtimes U)_q \otimes \mathcal{E}_m(V \boxtimes U)_q$ of $q$-coinvariants of the $\mathfrak{gl}_{m+1}$-module $\mathcal{E}_{m+1}(V \boxtimes U)$. This space is a $Y(\mathfrak{gl}_m)$-module, which also inherits the action of the Lie algebra $\mathfrak{gl}_m \oplus \mathfrak{gl}_1$. Our Theorem 2.1 states that the bimodule $\mathcal{E}_{m+1}(V \boxtimes U)_q \otimes \mathcal{E}_m(V \boxtimes U)_q$ over $Y(\mathfrak{gl}_m)$ and $\mathfrak{gl}_m \oplus \mathfrak{gl}_1$ is equivalent to the tensor product $\mathcal{E}_m(V) \otimes \mathcal{E}_m(U)$. Here we use the counit $\gamma : U(\mathfrak{gl}_m) \to \mathbb{C}$ on $\mathfrak{gl}_m$.

In Section 3 we propose a representation theoretic explanation of the correspondence between intertwining operators on the tensor products of certain $Y(\mathfrak{gl}_n)$-modules, and the “extremal cocycle” on the Weyl group $\mathfrak{S}_m$ of the reductive Lie algebra $\mathfrak{gl}_m$ as defined by K. Zhelobenko [Z]. This correspondence, discovered by V. Tarasov and A. Varchenko [TV2], was one of the motivations of our work. The arguments of [TV2], inspired by the results of V. Toledano-Laredo [T], are based on the classical duality theorem [H] which asserts that the images of $U(\mathfrak{gl}_m)$ and $U(\mathfrak{gl}_n)$ in the ring $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$ of differential operators on $\mathbb{C}^m \otimes \mathbb{C}^n$ with polynomial coefficients are the commutants of each other. Results relevant to this correspondence have also been obtained by Y. Smirnov and V. Tolstoy [ST]. Our explanation of this correspondence is based on the theory of Mickelsson algebras [M1], [M2] as developed in [KO]. Consider the tensor product

$$U(\mathfrak{gl}_m) \otimes \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n).$$

We have a representation $\gamma : U(\mathfrak{gl}_m) \to \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$. Taking the composition of the comultiplication map on $U(\mathfrak{gl}_m)$ with the homomorphism $\text{id} \otimes \gamma$ we get an embedding of $U(\mathfrak{gl}_m)$ into the algebra (0.1). Our particular Mickelsson algebra is determined by the pair formed by the algebra (0.1), and its subalgebra $U(\mathfrak{gl}_m)$ relative to this embedding. From another perspective, connections between the representation theory of the Yangian $Y(\mathfrak{gl}_m)$ and the theory of Mickelsson algebras have been studied by A. Molev [M].

Our results can be restated in the language of dynamical Weyl groups as used by P. Etingof and A. Varchenko in [EV]. However, our approach makes more natural the appearance of the Yangian $Y(\mathfrak{gl}_m)$ in the context of the classical dual pair $(\mathfrak{gl}_m, \mathfrak{gl}_n)$ of reductive Lie algebras. Moreover, results of the present article can be extended to other reductive dual pairs [H]. This will be done in our forthcoming publications.

We complete this article with an observation on the “centralizer construction” of the Yangian $Y(\mathfrak{gl}_n)$ due to G. Olshanski [O1]. For any two irreducible polynomial modules $V$ and $V'$ over the Lie algebra $\mathfrak{gl}_m$, the results of [O1] provide an action of $Y(\mathfrak{gl}_m)$ on the vector space

$$\text{Hom}_{\mathfrak{gl}_m}(V', V \otimes S(\mathbb{C}^m \otimes \mathbb{C}^n)).$$