Abstract. The Knop–Sahi interpolation Macdonald polynomials are inhomogeneous and nonsymmetric generalisations of the well-known Macdonald polynomials. In this paper we apply the interpolation Macdonald polynomials to study a new type of basic hypergeometric series of type $\mathfrak{gl}_n$. Our main results include a new $q$-binomial theorem, a new $q$-Gauss sum, and several transformation formulae for $\mathfrak{gl}_n$ series.

Part I. Interpolation Macdonald polynomials

1. Introduction

The Newton interpolation polynomials

$$N_k(x) = (x - x_1) \cdots (x - x_k) \quad (1.1)$$

were used by Newton in his now famous expansion

$$N(x) = \sum_{i=0}^{k} N(x_1) \partial_1 \partial_2 \cdots \partial_i N_i(x).$$
Here $N(x)$ is an arbitrary polynomial of degree $k$, and $\partial_i$ (operators in this paper act on the left) is a Newton divided difference operator

$$f(x_1, x_2, \ldots)\partial_i = \frac{f(\ldots, x_{i+1}, x_i, \ldots) - f(\ldots, x_i, x_{i+1}, \ldots)}{x_{i+1} - x_i}.$$ 

Various multivariable generalisations of the Newton interpolation polynomials exist in the literature, such as the Schubert polynomials [12] and several types of Macdonald interpolation polynomials [10], [11], [23], [24], [25], [29], [31]. In this paper we are interested in the latter, providing generalisations of (1.1) when the interpolation points $x_1, \ldots, x_k$ form a geometric progression

$$(x_1, x_2, x_3, \ldots) = (1, q, q^2, \ldots).$$

Then

$$N_k(x) = (x - 1)(x - q) \cdots (x - q^{k-1}),$$

and three equivalent characterisations may be given as follows:

1. $N_k(x)$ is the unique monic polynomial of degree $k$ such that $N_k(q^m) = 0$ for $m \in \{0, 1, \ldots, k - 1\}$.
2. $N_k(x)$ is the solution of the recurrence

$$p_{k+1}(x) = q^k(x - 1)p_k(x/q)$$

with initial condition $p_0(x) = 1$.
3. Up to normalisation $N_k(x)$ is the unique polynomial eigenfunction, with eigenvalue $q^{-k}$, of the operator

$$\xi = \tau \left(1 - \frac{1}{x}\right) + \frac{1}{x},$$

where $f(x)\tau = f(x/q)$.

Knop [10] and Sahi [29] generalised the Newton interpolation polynomials to a family of nonsymmetric, inhomogeneous polynomials $M_u(x)$, labelled by compositions $u \in \mathbb{N}^n$ and depending on $n$ variables; $x = (x_1, \ldots, x_n)$. The polynomials $M_u(x)$, known as the (nonsymmetric) Macdonald interpolation polynomials or (nonsymmetric) vanishing Macdonald polynomials, form a distinguished basis in the ring $\mathbb{Q}(q, t)[x_1, \ldots, x_n]$. Remarkably, Knop and Sahi showed that all three characterisations of the Newton interpolation polynomials carry over to the multivariable theory. What appears not to have been observed before, however, is that the Macdonald interpolation polynomials may be employed to build a multivariable theory of basic hypergeometric series of type $\mathfrak{gl}_n$. For example, with $M_u(x)$ an appropriate normalisation of $M_u(x)$, the following $n$-dimensional extension of the famous $q$-binomial theorem holds

$$\sum_u a^{|u|} M_u(x) = \prod_{i=1}^n \frac{(at^{n-i})_\infty}{(ax_i)_\infty}.$$  

(1.3)