THE ORIENTATION-PRESERVING
DIFFEOMORPHISM GROUP OF $\mathbb{S}^2$
DEFORMS TO SO(3) SMOOTHLY

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Abstract. Smale proved that the orientation-preserving diffeomorphism group of $\mathbb{S}^2$ has a continuous strong deformation retraction to SO(3). In this paper, we construct such a strong deformation retraction which is diffeologically smooth.

1. Introduction

In Smale’s 1959 paper “Diffeomorphisms of the 2-Sphere” ([10]), he shows that there is a continuous strong deformation retraction from the orientation-preserving $C^\infty$ diffeomorphism group of $\mathbb{S}^2$ to the rotation group SO(3). The topology of the former is the $C^k$ topology. In this paper, we construct such a strong deformation retraction which is diffeologically smooth. We follow the general idea of [10], but to achieve smoothness, some of the steps we use are completely different from those of [10]. The most notable difference is the use of Sobolev inequalities as opposed to general topological arguments. We note that there is a different proof of Smale’s result in [2], but the homotopy is not shown to be smooth.

The main result of this paper is as follows. Let us denote

\[ \Omega := \text{the orientation-preserving } C^\infty \text{ diffeomorphism group of } \mathbb{S}^2. \]

Also, let $x_0$ denote the South Pole on $\mathbb{S}^2$, and let

\[ \Omega_1 := \{ f \in \Omega \mid f(x_0) = x_0 \text{ and } df|_{x_0} = \text{id}_{T_{x_0}\mathbb{S}^2} \}. \]

In the following theorem, we contract $\Omega_1$ to $\{\text{id}_{\mathbb{S}^2}\}$.

Main Theorem. There is a smooth strong deformation retraction $R : I \times \Omega_1 \to \Omega_1$ to $\{\text{id}_{\mathbb{S}^2}\}$. More precisely, for each $(t, f) \in I \times \Omega_1$,

(1) $R_0(f) = f$;
(2) $R_1(f) = \text{id}_{\mathbb{S}^2}$;
(3) $R_t(\text{id}_{\mathbb{S}^2}) = \text{id}_{\mathbb{S}^2}$.

As a corollary, we have the following.

DOI: 10.1007/s00031-011-9130-0

Received May 15, 2010. Accepted January 26, 2011. Published online March 24, 2011.
Corollary. There is a smooth strong deformation retraction \( P : I \times \Omega \to \Omega \) that is equivariant under the left action of \( \text{SO}(3) \). More precisely, for each \( (t, f) \in I \times \Omega \) and \( A \in \text{SO}(3) \),

1. \( P_0(f) = f \);
2. \( P_1(f) \in \text{SO}(3) \);
3. \( P_t(A) = A \);
4. \( P_t(A \circ f) = A \circ P_t(f) \).

In order to prove the Main Theorem, we need to make use of a proposition about the diffeomorphisms of a square, which is of interest on its own. We follow Smale’s proof closely (see [10]). We state the proposition here. Let \( \mathcal{F} \) be the space of those orientation-preserving diffeomorphisms of the square \([-1, 1]^2\) such that for each \( f \in \mathcal{F} \), there exists a neighborhood of the boundary \( \partial([-1, 1]^2) \) on which \( f \) is the identity map.

Proposition 1.1. There is a smooth strong deformation retraction \( F : I \times \mathcal{F} \to \mathcal{F} \) to \( \{ \text{id}_{[-1,1]^2}\} \). More precisely, for each \( (t, f) \in I \times \mathcal{F} \),

1. \( F_0(f) = f \);
2. \( F_1(f) = \text{id}_{[-1,1]^2} \);
3. \( F_t(\text{id}_{[-1,1]^2}) = \text{id}_{[-1,1]^2} \).

This paper is organized as follows. In Section 2, we give a brief review of diffeology. In Section 3, we prove the Main Theorem while assuming Proposition 1.1. We break the homotopy up into smaller homotopies and smoothly concatenate them in the end. In Section 4 we prove the Corollary in a similar style, by breaking up the homotopy into smaller ones and smoothly concatenating them together. In Section 5, we prove Proposition 1.1.

Acknowledgements. We would like to express our deepest gratitude to Yael Karshon for her time and patience involved in supervising this project. We would also like to thank Katrin Wehrheim for a helpful suggestion in the proof of Lemma 3.1. Finally, we would like to thank the referees for giving valuable advice on improving the presentation of the paper.

2. Review of diffeological spaces

We start by defining the notion of diffeological smoothness in three special cases which are directly applicable to this paper. Note that diffeology can be defined in a much more general context and we refer the readers to [5].

Definition 2.1. Let \( U \) be an arbitrary open set in a Euclidean space of arbitrary dimension.

- Suppose \( \Lambda \) is a manifold with corners. A map \( P : U \to \Lambda \) is a plot if \( P \) is \( C^\infty \).
- Suppose \( X \) and \( Y \) are manifolds with corners, and \( \Lambda \subset C^\infty(X, Y) \). Denote by \( \text{ev} \) the evaluation map \( \Lambda \times X \to Y \) given by \( (f, x) \mapsto f(x) \). A map \( P : U \to \Lambda \)
  is a plot, if the map from \( U \times X \) to \( Y \) given by \( (s, x) \mapsto \text{ev}(P(s), x) \) is \( C^\infty \).