Abstract. Our base field is the field \( \mathbb{C} \) of complex numbers. We study families of reductive group actions on \( \mathbb{A}^2 \) parametrized by curves and show that every faithful action of a non-finite reductive group on \( \mathbb{A}^3 \) is linearizable, i.e., \( G \)-isomorphic to a representation of \( G \). The difficulties arise for non-connected groups \( G \).

We prove a Generic Equivalence Theorem which says that two affine morphisms \( p: S \to Y \) and \( q: T \to Y \) of varieties with isomorphic (closed) fibers become isomorphic under a dominant étale base change \( \phi: U \to Y \). A special case is the following result. Call a morphism \( \varphi: X \to Y \) a fibration with fiber \( F \) if \( \varphi \) is flat and all fibers are (reduced and) isomorphic to \( F \). Then an affine fibration with fiber \( F \) admits an étale dominant morphism \( \mu: U \to Y \) such that the pull-back is a trivial fiber bundle: \( U \times_Y X \simeq U \times F \).

As an application we give short proofs of the following two (known) results:

(a) Every affine \( \mathbb{A}^1 \)-fibration over a normal variety is locally trivial in the Zariski-topology (see [KW85]).

(b) Every affine \( \mathbb{A}^2 \)-fibration over a smooth curve is locally trivial in the Zariski-topology (see [KZ01]).

1. Introduction and main results

Linearization

Our base field is the field \( \mathbb{C} \) of complex numbers. For a variety \( X \) we denote by \( \mathcal{O}(X) \) the algebra of regular functions on \( X \), i.e., the global sections of the sheaf \( \mathcal{O}_X \) of regular functions on \( X \). An action of an algebraic group \( G \) on \( X \) is called linearizable if \( X \) is \( G \)-equivariantly isomorphic to a linear representation of \( G \). The “Linearization Problem” asks if every action of a reductive algebraic group \( G \) on affine \( n \)-space \( \mathbb{A}^n \) is linearizable. For \( n = 2 \) the problem has a positive answer, due to the structure of the automorphism group of \( \mathbb{A}^2 \) as an amalgamated product. On the other hand, there exist non-linearizable actions on certain \( \mathbb{A}^n \) for all non-
commutative connected reductive groups; see [Sch89], [Kno91]. The open cases are
commutative reductive groups, in particular, tori, and commutative finite groups. For a survey on this problem we refer to the literature ([Kra96], [KS92]).

A very interesting case is dimension 3, where no counterexamples have occurred so far. It is known that all actions of semisimple groups are linearizable ([KP85]) as well as $\mathbb{C}^*$-actions (see [KKMLR97]). The following result completes the picture of reductive group actions on $\mathbb{A}^3$.

**Theorem A.** Every faithful action of a non-finite reductive group on $\mathbb{A}^3$ is linearizable.

The remaining case of a finite group action on $\mathbb{A}^3$ seems to be a very difficult problem.

**Generic isotriviality**

One of the basic results of our paper is the following “generic isotriviality” of group actions (Theorem 2.2).

**Theorem B.** Let $\varphi: X \to Y$ be a dominant morphism where $X, Y$ are irreducible, and let $G$ be a reductive group acting on $X/Y$. Assume that the action of $G$ on the general fiber of $\varphi$ is linearizable. Then there is a dominant morphism of finite degree $\mu: U \to Y$ such that the fiber product $X \times_Y U$ is $G$ isomorphic to $W \times U$ over $U$ where $W$ is a linear representation of $G$:

$$
\begin{array}{cccc}
W \times U & \xrightarrow{\cong} & X \times_Y U & \longrightarrow & X \\
| & \downarrow \text{pr} & \downarrow & \downarrow \varphi \\
U & \longrightarrow & U & \xrightarrow{\mu} & Y \\
\end{array}
$$

As usual, the condition that “the action of $G$ on the general fiber of $\varphi$ is linearizable” means that on an open dense subset of $Y$ all fibers $\varphi^{-1}(y)$ are reduced and $G$-isomorphic to a representation of $G$. Clearly, if the characteristic of the base field is zero, then the dominant map $\mu: U \to Y$ can be chosen to be étale.

Theorem B is based on a very general result, the “Generic Equivalence Theorem” which we formulate and prove in section 2. Several special cases of this result appear in the literature, quite often in connection with so-called “cylinder-like open sets”, but the statement seems not to be known in this general form.

In the last paragraph we use this result to give a short and unified proof of the following results due to Kambayashi–Wright and Kaliman–Zaidenberg (Theorem 5.2).

**Theorem C.**

(a) If $\varphi: X \to Y$ is a flat affine morphism with fibers $\mathbb{A}^1$ and $Y$ normal, then $\varphi$ is a fiber bundle, locally trivial in the Zariski-topology.

(b) If $\varphi: X \to Y$ is a flat affine morphism with fibers $\mathbb{A}^2$ and $Y$ a smooth curve, then $\varphi$ is a fiber bundle, locally trivial in the Zariski-topology.