ON REDUCTIVE AUTOMORPHISM GROUPS OF
REGULAR EMBEDDINGS

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Abstract. Let $G$ be a connected reductive complex algebraic group acting on a smooth complete complex algebraic variety $X$. We assume that $X$ is a regular embedding, a condition satisfied in particular by smooth toric varieties and flag varieties. For any set $\mathcal{D}$ of $G$-stable prime divisors, we study the action on $X$ of the group $\text{Aut}^\circ(X, \mathcal{D})$, the connected automorphism group of $X$ stabilizing all elements of $\mathcal{D}$. We determine a Levi subgroup $A(X, \mathcal{D})$ of $\text{Aut}^\circ(X, \mathcal{D})$, and also relevant invariants of $X$ as a spherical $A(X, \mathcal{D})$-variety. As a byproduct, we obtain a complete description of the inclusion relation between closures of $A(X, \mathcal{D})$-orbits on $X$.

1. Introduction

In the 1970s Michel Demazure described the connected automorphism groups of two distinguished classes of algebraic varieties equipped with the action of a connected reductive group $G$: complete homogeneous spaces (see [De77]), and smooth complete toric varieties (see [De70]).

These two classes of $G$-varieties admit a common generalization: the regular embeddings, here also called $G$-regular embeddings, defined independently in [BDP90] and [Gi89]. With the additional assumption of completeness, Frédéric Bien and Michel Brion showed that these varieties form a relevant class of spherical varieties, which are by definition normal $G$-varieties with a dense orbit of a Borel subgroup of $G$.

The goal of this paper is to study the connected automorphism group $\text{Aut}^\circ(X)$ of a complete regular embedding $X$. More precisely, we are interested in the group $\text{Aut}^\circ(X, \mathcal{D})$, where $\mathcal{D}$ is any set of $G$-stable prime divisors of $X$, and $\text{Aut}^\circ(X, \mathcal{D})$ is the connected automorphism group of $X$ stabilizing all elements of $\mathcal{D}$.

Our results are divided into several steps according to additional hypotheses on $\mathcal{D}$ and on $G$, and in each step we accomplish two goals. The first one is to describe
a Levi subgroup $A(X, D)$ of $\text{Aut}^\circ(X, D)$, based on the knowledge of discrete invariants associated with the $G$-action of $X$. These invariants come from the theory of spherical varieties, and are: the group $\Delta_G(X)$ of $B$-eigenvalues of $B$-eigenvectors of $C(X)$, the set of spherical roots $\Sigma_G(X)$ of $X$ (see Definition 2.4), the set $\Delta_G(X)$ of $B$-stable but not $G$-stable prime divisors of $X$, and the stabilizer $P_G(X)$ of the open $B$-orbit of $X$. The last invariant is the fan $\mathcal{F}_G(X)$, a collection of strictly convex polyhedral cones in the vector space $\text{Hom}_\mathbb{Z}(\Delta_G(X), \mathbb{Q})$ (see Definition 2.3). The fan determines $X$ uniquely among the complete regular embeddings having the same open $G$-orbit, generalizing the fan associated with a toric variety. It also provides a combinatorial description of the $G$-orbits of $X$ and of the inclusion relation between $G$-orbit closures (see [Kn91]).

Now $X$ is a spherical variety also under the action of $A(X, D)$, and our second goal is to determine the above invariants of $X$ with respect to the action of $A(X, D)$. In particular, this provides a combinatorial description of the $A(X, D)$-orbits on $X$.

Our approach is based on the analysis of a spherical variety $X$ canonically associated with $X$ and equipped with a canonical $G$-equivariant map $X \to X$. The variety $X$, defined in Section 3, is obtained from $X$ using a procedure called wonderful closure, which is closely related to the well-known construction of the spherical closure of a generic stabilizer of a spherical variety. We introduce the wonderful closure because it turns out to give much more direct information on the automorphisms of $X$ than the spherical closure. On the other hand, wonderful varieties such as $X$ (see Definition 3.3) play a central role in the theory of spherical varieties (see, e.g., [Lu01]), and their automorphism groups have already been studied in [Br07] and [Pe09].

Our study of the group $\text{Aut}^\circ(X, D)$ proceeds by “successive approximation” with a sequence of subgroups that starts with elements very closely related to $\text{Aut}^\circ(X)$. More precisely, we consider the filtration

$$\text{Aut}^\circ(X, \partial X) \subseteq \text{Aut}^\circ(X, D \cup (\partial X)^\ell) \subseteq \text{Aut}^\circ(X, D),$$

where we denote by $\partial X$ the set of all $G$-stable prime divisors of $X$ and by $(\partial X)^\ell$ the subset of $G$-invariant prime divisors mapping surjectively onto $X$.

The group $\text{Aut}^\circ(X, \partial X)$ is also the connected group of automorphisms stabilizing all $G$-orbits, and is the subject of Section 4. This group is reductive after [Br07] and $X$ is regular under its action; moreover, $\text{Aut}^\circ(X, \partial X)$ is equal to $\text{Aut}^\circ(X, \partial X)$ up to central isogeny and up to a torus factor. We recall that $\text{Aut}^\circ(X, D)$ for any $D \subseteq \partial X$ is reductive, and known after [Pe09].

After providing some technical results relating the automorphisms of $X$ and $X$ in Sections 5 and 6, we devote Section 7 to $\text{Aut}^\circ(X, D \cup (\partial X)^\ell)$. We show that this group is again reductive: it is equal to $\text{Aut}^\circ(X, D \cup (\partial X)^\ell)$ up to central isogeny and up to a torus factor, where $D \subseteq \partial X$ is now the set of the images $\pi(D) \in \partial X$ such that $D \in \partial X$ is stable under $\text{Aut}^\circ(X, D \cup (\partial X)^\ell)$.

This is the most technically involved part of the paper, and we use an indirect approach. First we analyze how the invariants of $X$ behave whenever $\text{Aut}^\circ(X, D \cup (\partial X)^\ell)$ is strictly bigger than $\text{Aut}^\circ(X, \partial X)$. Under this hypothesis the fan $\mathcal{F}_G(X)$ has a particularly nice structure and is essentially determined by