Sharp parameter ranges in the uniform anti-maximum principle for second-order ordinary differential operators

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To L.E. Payne on the occasion of his 80th birthday

Abstract. We consider the equation \((pu')' - qu + \lambda wu = f\) in \((0,1)\) subject to homogenous boundary conditions at \(x = 0\) and \(x = 1\), e.g., \(u'(0) = u'(1) = 0\). Let \(\lambda_1\) be the first eigenvalue of the corresponding Sturm-Liouville problem. If \(f \leq 0\) but \(\neq 0\) then it is known that there exists \(\delta > 0\) (independent on \(f\)) such that for \(\lambda \in (\lambda_1, \lambda_1 + \delta]\) any solution \(u\) must be negative. This so-called uniform anti-maximum principle (UAMP) goes back to Clément, Peletier [4]. In this paper we establish the sharp values of \(\delta\) for which (UAMP) holds. The same phenomenon, including sharp values of \(\delta\), can be shown for the radially symmetric \(p\)-Laplacian on balls and annuli in \(\mathbb{R}^n\) provided \(1 \leq n < p\). The results are illustrated by explicitly computed examples.

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1. Introduction and main results

Let \(L = a_{ij}(x)\partial_{ij}^2 + b_i(x)\partial_i + c(x)\) be a uniformly elliptic operator on a bounded \(C^2\)-domain \(\Omega \subset \mathbb{R}^n\) with continuous coefficients. Consider the boundary value problem

\[
Lu + \lambda u = f \text{ in } \Omega, \quad \partial_{\nu} u + bu = 0 \text{ on } \partial \Omega
\]

with a \(C^1\)-function \(b \geq 0\) and \(f \in L^q(\Omega), q > n\). Let \(\lambda_1\) denote the first eigenvalue of \(L\) subject to the above boundary condition. Then two important principles hold for solutions \(u \in W^{2,q}(\Omega)\) of (1):

Maximum principle (MP): If \(f \leq 0, f < 0\) on a set of positive measure and \(\lambda < \lambda_1\) then \(u > 0\) in \(\Omega\).

Anti-maximum principle (AMP): If \(f \leq 0, f < 0\) on a set of positive measure then there exists \(\delta = \delta(f) > 0\) such that \(\lambda \in (\lambda_1, \lambda_1 + \delta]\) implies \(u < 0\) in \(\Omega\).

The (AMP) was discovered by Clément, Peletier [4]. In the same paper a proof
of the well known (MP) is given. In [4] the authors also consider the uniform anti-
maximum principle (UAMP), where the constant $\delta$ does not depend on the data $f$. They showed that (UAMP) holds in dimension $n = 1$ (and does not hold in higher dimensions). For example, by computing the Green function $G(s, t)$ for the operator $d^2/dx^2 + \lambda$ on the interval $(0, 1)$ with Neumann boundary conditions at $x = 0, 1$ one finds that $G(s, t) < 0$ for $\lambda \in (0, \pi^2/4)$, whereas $G(s, t)$ is sign-changing for $\lambda > \pi^2/4$. Hence for
$$u'' + \lambda u = f \text{ in } (0, 1), \quad u'(0) = u'(1) = 0$$
(UAMP) holds precisely for $\lambda \in (0, \pi^2/4)$.

Subsequently, both (AMP) and (UAMP) have been extended to linear problems
with sign-changing weight by Hess [9] and Godoy et al. [8]. Recently Clément and
Sweers [5] found conditions under which (AMP) and (UAMP) hold for higher-
order elliptic boundary value problems. For the $p$-Laplacian $\text{div}(\nabla u^{p-2}\nabla u) + qu + lambda u = f$ in $(0, 1)$, $u(0) = u(1) = 0$ (UAMP) holds precisely for $\lambda \in (0, \pi^2/4)$.

In this paper we characterize the precise parameter-range of (UAMP) for the
problem
$$(pu')' - qu + \lambda wu = f \text{ in } (0, 1), \quad Bu = 0,$$  \hspace{1cm} (2)
where $Bu$ is an abbreviation for the boundary condition
$$(au + pu')|_{x=0} = 0, \quad (\beta u + pu')|_{x=1} = 0$$
with constants $\alpha, \beta \in \mathbb{R}$. We assume $p, q, w, f \in C[0, 1]$ and $p, w > 0$ in $[0, 1]$. Solutions are understood such that $u, pu' \in C^1[0, 1]$. To describe our results we use the following notation: let $\lambda^{\alpha \beta}_1$, $\lambda^{\infty \beta}_1$ stands for the first eigenvalue with zero Dirichlet boundary condition at $x = 0$, $x = 1$, respectively, and unchanged boundary condition at the opposite endpoint, i.e.,
$$\lambda^{\alpha \beta}_1: \quad (au + pu')|_{x=0} = 0, \quad (\beta u + pu')|_{x=1} = 0,$$
$$\lambda^{\infty \beta}_1: \quad u(0) = 0, \quad (\beta u + pu')|_{x=1} = 0,$$
$$\lambda^{\alpha \infty}_1: \quad (au + pu')|_{x=0} = 0, \quad u(1) = 0.$$  The corresponding eigenfunctions are denoted by $u^{\alpha \beta}_1$, $u^{\infty \beta}_1$ and $u^{\alpha \infty}_1$.

**Theorem 1.** (i) (UAMP) holds for (2) if $\lambda \in (\lambda^{\alpha \beta}_1, \min\{\lambda^{\infty \beta}_1, \lambda^{\alpha \infty}_1\})$ and $f \leq 0$, $f \not\equiv 0$ with the conclusion $u < 0$ in $[0, 1]$. (ii) For every $\lambda > \min\{\lambda^{\infty \beta}_1, \lambda^{\alpha \infty}_1\}$ there exists a function $f \leq 0$ and a sign-changing solution $u$ of (2).