A note on the small-time development of the solution to a scalar, non-linear, singular reaction-diffusion equation

P. M. McCabe∗, J. A. Leach and D. J. Needham

Abstract. In this note, we consider a class of scalar, non-linear, singular (in the sense that the reaction terms in the equation are not Lipschitz continuous) reaction-diffusion equations with positive initial data being of (a) $O(x^{-\alpha})$ or (b) $O(x^{-\beta}e^{-\sigma x})$ at large $x$ (dimensionless distance), where $\alpha, \sigma > 0$ and $\beta$ are constants. We establish, by developing the small-time asymptotic structure of the solution, that the support of the solution becomes finite in infinitesimal time in both cases (a) and (b) above. The asymptotic form for the location of the edge of the support as $t \to 0$ is given in both cases.

Keywords. Reaction-diffusion, small-time asymptotics.

1. Introduction

In this note we consider the following initial-boundary value problem for a scalar, non-linear, singular (i.e. non-Lipschitz) reaction-diffusion equation, namely,

$$
\begin{aligned}
&\begin{cases}
  u_t = u_{xx} + (1 - u)u^n - ku^n, & x, t > 0, \\
  u(x, 0) = u_0(x), & x \geq 0, \\
  u_x(0, t) = 0, & t > 0, \\
  u(x, t) \to 0 & \text{as } x \to \infty, \\
  \end{cases} \\
&\text{[P, m, n]}
\end{aligned}
$$

where $0 < n < m < 1$, $k > 0$ and $u_0(x)$ is a continuous, analytic, positive and monotone decreasing function in $x \geq 0$, with $u_0(x) \to 0$ as $x \to \infty$. In particular, we consider the following cases

(a) Initial data that has algebraic decay rate as $x \to \infty$

$$

u_0(x) \sim \begin{cases} u_\infty x^{-\alpha} + EST(x) & \text{as } x \to \infty, \\
 u_0 + \sum_{l=1}^{\infty} \tilde{u}_l x^l & \text{as } x \to 0^+, \end{cases}
$$

where $\alpha, u_\infty, \tilde{u}_0 > 0$, $\tilde{u}_l$ are constants and $EST(x)$ denotes exponentially small terms in $x$ as $x \to \infty$.

∗Current Address: ABN AMRO Bank N.V
(b) Initial data that has exponential decay rate as $x \to \infty$

\[ u_0(x) \sim \begin{cases} u_\infty x^{-\beta} e^{-\sigma x} + O(e^{-f(x)}) & \text{as } x \to \infty, \\ \tilde{u}_0 + \sum_{l=1}^{\infty} \tilde{u}_l x^l & \text{as } x \to 0^+, \end{cases} \tag{1.2} \]

for some $f(x) > O(x)$ as $x \to \infty$, where $u_\infty, \tilde{u}_0, \sigma > 0$ and $\beta, \tilde{u}_l$ are constants.

The uniqueness of the solution to $[P,m,n]$ with $n < m < 1$ follows directly from the comparison theorem (see McCabe, Leach and Needham [8]), whilst global existence follows after minor modifications to the results of Bandle and Stakgold [2].

The initial-boundary value problem $[P,m,n]$ arises as a scalar approximation to a system of singular reaction-diffusion equations which model an isothermal, autocatalytic chemical reaction with termination. The scheme is represented formally by the two steps,

\[ A \to B, \quad \text{rate } k_1 a b^m, \quad (\text{autocatalysis}), \tag{1.3} \]
\[ B \to C, \quad \text{rate } k_2 b^n, \quad (\text{decay}), \tag{1.4} \]

where $a$ and $b$ are the concentrations of the reactant $A$ and the autocatalyst $B$ respectively, $k_1 > 0$ is the constant rate of autocatalysis, $k_2 > 0$ is the constant rate at which the autocatalyst $B$ decays to the inert, stable product $C$, and $m$ and $n$ are the (fractional, $0 < m, n < 1$) orders of autocatalysis and decay. The dependent variable $u$ in $[P,m,n]$ may be regarded as an approximation to $b$ when $k_2 \ll k_1 a_0^{m-n+1}$ (we note that the parameter $k$ is defined by $k = k_2/k_1 a_0^{m-n+1}$) where $a_0$ is the initial concentration of $A$. A full description of the chemical model, its analysis and a comprehensive review of the relevant literature may be found in [7]. Preliminary results concerning the system of singular reaction-diffusion equations can be found in [4] and [6]. The initial-boundary value problem $[P,m,n]$ with initial data $u_0(x)$ having compact support has been considered in detail in [7] and [8].

In this note we use the method of matched asymptotic expansions to develop the small-time asymptotic structure of the solution to $[P,m,n]$ with $0 < n < m < 1$ when the initial data has algebraic or exponential decay rates as $x \to \infty$, given by (1.1) and (1.2) respectively. Throughout we use the nomenclature of the theory of matched asymptotic expansions, as given in Van Dyke [10] (see also Hinch [3], Lagerstrom [5] and Nayfeh [9] for an introduction to the theory of matched asymptotic expansions). We establish that in both cases the support of the solution becomes finite in infinitesimal time. We conclude by presenting the asymptotic form for the location of the edge of the support as $t \to 0$ in both cases. We further note that the asymptotic structure as $t \to 0$ is independent of the parameter $m$ in the orders considered here (provided $n < m$). However, consideration of higher order terms leads to the inclusion of terms involving the parameter $m$. 