NOTES ON FADING-MEMORY CONDITIONS*

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Abstract. In some studies concerning the approximation of nonlinear systems, the concept of a weighting function \( w \) plays a central role. This paper, in which attention is focused on continuous-time cases, directs attention to some interesting properties of systems that have \( \mathbb{R}_+ \) fading memory or fading memory. In particular, we show that half-line input-output maps that have \( \mathbb{R}_+ \) fading memory with respect to some \( w \) in fact have \( \mathbb{R}_+ \) fading memory with respect to all such \( w \)’s. And we show that a similar proposition holds in a setting concerning Volterra-series approximations for continuous-time systems with inputs and outputs defined on all of \( \mathbb{R} \). We show also that, in that setting, fading memory is equivalent to uniform fading memory. Some related results are also described.

Key words: \( \mathbb{R}_+ \) fading memory, uniform fading memory, fading memory, approximately finite memory, input-output maps, nonlinear systems.

1. Introduction

The concepts of maps that are myopic, or that have approximately finite memory or fading memory, play a central role in several papers (see, e.g., [1], [6], [8], [10]–[15], [18]) that consider general properties of nonlinear input-output maps and, in particular, address aspects of the problem of uniformly approximating input-output maps using certain simple nonlinear structures. In these papers, one finds results for continuous-time systems with inputs and outputs defined on either \( \mathbb{R}_+ \) or \( \mathbb{R} \), as well as results for discrete-time systems with inputs and outputs defined on either the nonnegative integers \( \mathbb{Z}_+ \) or the integers \( \mathbb{Z} \). There are also results for cases in which the inputs and outputs depend on more than one
For the case of discrete-time maps $G$ taking a certain set $S$ of bounded inputs defined on the nonnegative integers $\mathbb{Z}_+$ into outputs on $\mathbb{Z}_+$, by $G$ having approximately finite memory we mean in [13] that for each $\gamma > 0$, there is a $\Delta > 0$ such that

$$|(Gs)(k) - (GW_{k,\alpha}s)(k)| < \gamma, \quad k \geq 0$$

for all $s \in S$ and integer $\alpha \geq \Delta$. Here $W_{k,\alpha}$ is the “window map” defined by

$$(W_{k,\alpha}s)(\tau) = s(\tau), \quad k - \alpha \leq \tau \leq k = 0, \quad \text{otherwise}.$$ 

Much is known about time-invariant systems that possess this property. For example, they can be approximated arbitrarily well by the maps of certain simple structures such as lattice map structures, finite Volterra-series structures, dynamic multilayered neural networks with sigmoidal hidden units, and dynamic radial basis function networks. An important fact that gives meaning to the results that have been established concerning approximately finite memory systems is that condition (1) is known to be often met. More specifically, in [13] (see also [2]) an example is given of an important large class of feedback systems for which (1) holds. The main assumption is that the circle criterion for stability is met.

The concept of fading memory in [1] for discrete-time systems is fundamentally different in that it addresses the case of causal time-invariant systems with inputs and outputs both defined on $\mathbb{Z}$, not $\mathbb{Z}_+$. There $H$ defined on the space $\ell_\infty$ of real-valued bounded inputs defined on $\mathbb{Z}$, and mapping $\ell_\infty$ into itself, has fading memory on a subset $V$ of $\ell_\infty$ if there exists a decreasing $w : \mathbb{Z}_+ \to (0, 1]$ with $\lim_{k \to \infty} w(k) = 0$ such that for each $v_a \in V$ and each $\epsilon > 0$ there is a $\delta > 0$ such that

$$|(Hv_a)(0) - (Hv_b)(0)| < \epsilon$$

whenever $v_b \in V$ and

$$\sup_{m \leq 0} w(-m)|v_a(m) - v_b(m)| < \delta.$$ 

A theorem is given in [1] to the effect that if $H$ has fading memory on $V$, then $H$ can be approximated arbitrarily well by a finite discrete-time Volterra-series operator. This is an interesting proposition but, to the best of this writer’s knowledge and prior to the appearance of [16], the literature did not contain results

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2 The concept of a “myopic” map (see, e.g., [12], [18]) is useful in connection with systems that need not be causal, as in image processing, because the terms “approximately finite memory” and “fading memory” are inappropriate when applied to noncausal systems, in that noncausal systems may anticipate as well as remember. Roughly speaking, an input-output map $M$ is myopic if the value of $(Mu)(\gamma)$ is always relatively independent of the values of $u$ at points remote from $\gamma$. The concepts of maps that are myopic, have approximately finite memory, fading memory, or decaying memory [10] are all different but are all related in that they are alternative ways of making precise, in different settings, the same general idea. For early work related in a general sense, see [3]–[5].