TENSOR SURGERY 
AND TENSOR RANK

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Abstract. We introduce a method for transforming low-order tensors into higher-order tensors and apply it to tensors defined by graphs and hypergraphs. The transformation proceeds according to a surgery-like procedure that splits vertices, creates and absorbs virtual edges and inserts new vertices and edges. We show that tensor surgery is capable of preserving the low rank structure of an initial tensor decomposition and thus allows to prove nontrivial upper bounds on tensor rank, border rank and asymptotic rank of the final tensors. We illustrate our method with a number of examples. Tensor surgery on the triangle graph, which corresponds to the matrix multiplication tensor, leads to nontrivial rank upper bounds for all odd cycle graphs, which correspond to the tensors of iterated matrix multiplication. In the asymptotic setting we obtain upper bounds in terms of the matrix multiplication exponent $\omega$ and the rectangular matrix multiplication parameter $\alpha$. These bounds are optimal if $\omega$ equals two. We also give examples that illustrate that tensor surgery on general graphs might involve the absorption of virtual hyperedges and we provide an example of tensor surgery on a hypergraph. Besides its relevance in algebraic complexity theory, our work has applications in quantum information theory and communication complexity.

Keywords. tensor rank, graph tensors, algebraic complexity, matrix multiplication

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1. Introduction

This paper introduces a method for proving upper bounds on tensor rank, border rank and asymptotic tensor rank. The method
gives particularly clean results when applied to tensors that are defined combinatorially. Let us first illustrate the combinatorial description that we are using and illustrate the method.

1.1. Illustration. The most famous example of a tensor that fits into our combinatorial framework (and which plays an important role in this paper) is the two-by-two matrix multiplication tensor, which is the 3-tensor described by the triangle graph $C_3$

\[
T_2\left(\begin{array}{c}
\end{array}\right) = \sum_{i \in \{0,1\}^3} (b_{i_1} \otimes b_{i_2}) \otimes (b_{i_2} \otimes b_{i_3}) \otimes (b_{i_3} \otimes b_{i_1}) \\
\in (\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes 3}
\]

where $\{b_0, b_1\}$ is the standard basis of $\mathbb{C}^2$. For a space $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$ of $k$-tensors we refer to the $\mathbb{C}^{n_i}$ as the tensor legs. Informally, the graph–tensor correspondence is as follows: each vertex of the graph corresponds to a tensor leg and each edge in the graph corresponds to an index to sum over, shared between tensor legs (see Section 2 for a formal definition). By default we view the above tensor as a 3-tensor, but we will sometimes view it as a 6-tensor. Another important example is the so-called rank-two unit 3-tensor, which corresponds to the hypergraph on three vertices with a single hyperedge $\{1, 2, 3\}$

\[
T_2\left(\begin{array}{c}
\end{array}\right) = \sum_{i \in \{0,1\}} b_i \otimes b_i \otimes b_i \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2.
\]

In the algebraic complexity theory literature, the two-by-two matrix multiplication tensor is usually denoted by $\langle 2, 2, 2 \rangle$ and the rank-two unit 3-tensor by $\langle 2 \rangle$. As a final illustrative example consider the complete graph on 4 vertices $K_4$ and the corresponding 4-tensor

\[
T_2\left(\begin{array}{c}
\end{array}\right) = \sum_{i \in \{0,1\}^6} (b_{i_1} \otimes b_{i_2} \otimes b_{i_3}) \otimes (b_{i_3} \otimes b_{i_4} \otimes b_{i_5}) \\
\otimes (b_{i_4} \otimes b_{i_5} \otimes b_{i_6}) \otimes (b_{i_1} \otimes b_{i_5} \otimes b_{i_6}) \\
\in (\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes 4}.
\]