RATIONAL FUNCTIONS WITH PRESCRIBED CRITICAL POINTS

I. Scherbak

Abstract
A rational function is the ratio of two complex polynomials in one variable without common roots. Its degree is the maximum of the degrees of the numerator and the denominator. Rational functions belong to the same class if one turns into the other by postcomposition with a linear-fractional transformation. We give an explicit formula for the number of classes having a given degree $d$ and given multiplicities $m_1, \ldots, m_n$ of given $n$ critical points, for generic positions of the critical points. This number is the multiplicity of the irreducible $sl_2$ representation with highest weight $2d - 2 - m_1 - \cdots - m_n$ in the tensor product of the irreducible $sl_2$ representations with highest weights $m_1, \ldots, m_n$.

The classes are labeled by the orbits of critical points of a remarkable symmetric function which first appeared in the XIX century in studies of Fuchsian differential equations, and then in the XX century in the theory of KZ equations.

1 Introduction

A rational function is the ratio of two complex polynomials in one variable without common roots. Its degree is the maximum of the degrees of the numerator and the denominator. Any rational function defines a branched covering of the Riemann sphere, $\mathbb{CP}^1 = \mathbb{C} \cup \infty$, and the degree is the cardinality of the preimage of any regular value.

Let $R(x) = g(x)/f(x)$, where $g(x)$ and $f(x)$ are polynomials without common roots, be a rational function of degree $d$. We have $R'(x) = W(x)/f^2(x)$, where

$$W(x) = W[g, f](x) = g'(x)f(x) - g(x)f'(x)$$

is the Wronskian of polynomials $g$ and $f$.

The rational function $R(x)$ has a critical point of multiplicity $m \geq 1$ at $z \in \mathbb{C}$, if $z$ is a root of the Wronskian $W(x)$ of multiplicity $m$. Let $m_1, \ldots, m_n$ be positive integers and $z_1, \ldots, z_n$ pairwise distinct complex numbers. Write $m = (m_1, \ldots, m_n)$, $M = m_1 + \cdots + m_n$ and $z = (z_1, \ldots, z_n)$. 
If \( R(x) \) has degree \( d > 1 \) and if all its finite critical points are \( z_1, \ldots, z_n \) of multiplicities \( m_1, \ldots, m_n \) respectively, then we say that \( R(x) \) has the type \((d, n; m; z)\). According to the Riemann–Hurwitz formula, numbers \( d \) and \( M \) should satisfy \( 2d - 2 \geq M \). The difference \( m_\infty = 2d - 2 - M \) is the multiplicity of \( R(x) \) at infinity.

If \( R(x) \) has the type \((d, n; m; z)\), then the rational function given by the postcomposition with a linear-fractional transformation,

\[
\frac{aR(x) + b}{cR(x) + d}, \quad ad - bc \neq 0,
\]

has clearly the same type. A rational function considered up to post-compositions with linear-fractional transformations will be called a class of rational functions.

We address the following question.

**Given** \( d, n, m, z \), **how many classes have the type** \((d, n; m; z)\)?

The question is easy for \( n = 1 \). If \( z \in \mathbb{C} \) is the single critical point, then the number of classes is 1 if the multiplicity of \( z \) is \( d - 1 \) (this is the class containing \((x - z)^d\)), and 0 otherwise.

If \( n \geq 2 \), we answer the question for generic \( z \). The words “\( z \) is generic” mean that \( z \) does not belong to a suitable proper algebraic set in \( \mathbb{C}^n \).

**Theorem.** Let \( n \geq 2 \) and \( d, m_1, \ldots, m_n \) be positive integers. For generic \( z \), the number \( \sharp(d, n; m) \) of classes of rational functions of the type \((d, n; m; z)\) is

\[
\sharp(d, n; m) = \sum_{q=1}^{n} (-1)^{n-q} \sum_{1 \leq i_1 < \cdots < i_q \leq n} \left( m_{i_1} + \cdots + m_{i_q} + q - d - 1 \right) \binom{n - 2}{n - 2}, \tag{1}
\]

and any nonempty class can be represented by the ratio of polynomials without multiple roots.

As usually we set \( \binom{a}{b} = 0 \) for \( a < b \).

**Remarks.**

- If \( m_j > d - 1 \) for some \( j \), or if \( M > 2d - 2 \), or if \( M < d - 1 \), then the right-hand side of (1) vanishes. In fact, in this case for any \( z \) there are no rational functions of the type \((d, n; m; z)\). Indeed, any of the inequalities is impossible for a rational map of degree \( d \). This was known, see [GrH, Ch.6] or [Go, Lemma 1.1].
- If \( M = d - 1 \) and \( m_j \leq d - 1 \) for \( j = 1, \ldots, n \), then the right hand side of (1) gives 1. Again, for any \( z \) the number of classes is 1. Indeed, if \( R(x) = g(x)/f(x) \) belongs to a corresponding class, and if \( g \) has