SHARP RATE OF AVERAGE DECAY OF THE FOURIER TRANSFORM OF A BOUNDED SET

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Abstract

Estimates for the decay of Fourier transforms of measures have extensive applications in numerous problems in harmonic analysis and convexity including the distribution of lattice points in convex domains, irregularities of distribution, generalized Radon transforms and others. Here we prove that the spherical $L^2$-average decay rate of the Fourier transform of the Lebesgue measure on an arbitrary bounded convex set in $\mathbb{R}^d$ is

$$\left( \int_{S^{d-1}} |\hat{\chi}_B(R\omega)|^2 \, d\omega \right)^{1/2} \lesssim R^{-\frac{d+1}{2}}.$$  \hfill (\ast)

This estimate is optimal for any convex body and in particular it agrees with the familiar estimate for the ball. The above estimate was proved in two dimensions by Podkorytov, and in all dimensions by Varchenko under additional smoothness assumptions. The main result of this paper proves (\ast) in all dimensions under the convexity hypothesis alone. We also prove that the same result holds if the boundary of $\partial \Omega$ is $C^{3/2}$.

Introduction

Let $B$ be a bounded open set in $\mathbb{R}^d$. If $\partial B$ is sufficiently smooth and has everywhere non-vanishing Gaussian curvature, then

$$|\hat{\chi}_B(R\omega)| \lesssim R^{-\frac{d+1}{2}},$$  \hfill (0.1)

with constants independent of $\omega$, where

$$f(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) \, dx,$$

denotes the Fourier transform, and $A \lesssim B$ means that there exists a positive constant $C$ such that $|A| \leq C|B|$. The estimate (0.1) is optimal in a very

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strong sense. One can check that a better rate of decay at infinity is not possible. One can also check that if the Gaussian curvature vanishes at even a single point, then (0.1) does not hold.

In fact, the point-wise estimate may be much worse. For example, if $B$ is convex, one has

$$\left| \hat{\chi}_B(R\omega) \right| \lesssim R^{-1},$$

and the case of a cube $[0, 1]^d$ shows that one cannot, in general, do any better. See, for example, [St2], for a nice description of these classical results.

In spite of the fact that the estimate (0.1) does not hold in general, a basic question is whether this estimate holds on average for a large class of domains, for example, bounded open sets with a rectifiable boundary. More precisely, one should like to know for which domains one has the following estimate:

$$\left( \int_{S^{d-1}} \left| \hat{\chi}_B(R\omega) \right|^2 d\omega \right)^{1/2} \lesssim R^{-\frac{d+1}{2}}.$$  \hspace{1cm} (0.2)

In some cases, it is equally useful to know whether

$$\left( \int_{S^{d-1}} \left| \hat{\sigma}(R\omega) \right|^2 d\omega \right)^{1/2} \lesssim R^{-\frac{d-1}{2}},$$  \hspace{1cm} (0.3)

where $\sigma$ is the Lebesgue measure on the boundary of $B$. Under a variety of assumptions, for example, if $\partial B$ is Lipschitz, (0.2) and (0.3) are linked via the divergence theorem. We use this fact in the proof of our main result below.

An example due to Sjölin ([Sj]) shows that (0.3) is not purely dimensional. He showed that if $\sigma$ is an arbitrary $(d - 1)$-dimensional compactly supported measure, then the best exponent one can expect on the right-hand side of (0.3) is $d - (3/2)$. This means that in order to prove an estimate like (0.2) we must use the fact that $\partial B$ is in some sense a hyper-surface.

Several results of this type have been proved over the years. In [P], Podkorytov proved (0.2) for convex domains in two dimensions using a beautiful geometric argument that relied on the fact that in two dimensions, the Fourier transform of a characteristic function of a convex set in a given direction is bounded by a measure of a certain geometric cap. See, for example, [BruNW] or [BrRT] for more details. Unfortunately, in higher dimensions one cannot bound the Fourier transform of a characteristic function of a convex set by such a geometric quantity. See, for example, [BMVW]. For the case of average decay on manifolds of co-dimension greater than one, see e.g. [Ch], [M], and [IS].