GRAPH PRODUCTS, FOURIER ANALYSIS AND SPECTRAL TECHNIQUES

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Abstract. We consider powers of regular graphs defined by the weak graph product and give a characterization of maximum-size independent sets for a wide family of base graphs which includes, among others, complete graphs, line graphs of regular graphs which contain a perfect matching and Kneser graphs. In many cases this also characterizes the optimal colorings of these products.

We show that the independent sets induced by the base graph are the only maximum-size independent sets. Furthermore we give a qualitative stability statement: any independent set of size close to the maximum is close to some independent set of maximum size.

Our approach is based on Fourier analysis on Abelian groups and on Spectral Techniques. To this end we develop some basic lemmas regarding the Fourier transform of functions on \(\{0, \ldots, r-1\}^n\), generalizing some useful results from the \(\{0,1\}^n\) case.

1 Introduction

Consider the following combinatorial problem:

Assume that at a given road junction there are \(n\) three-position switches that control the red-amber-green position of the traffic light. You are told that whenever you change the position of all the switches then the color of the light changes. Prove that in fact the light is controlled by only one of the switches.

The above problem is a special case of the problem we wish to tackle in this paper, characterizing the optimal colorings and maximal independent...
sets of products of regular graphs. The configuration space of the switches described above can be modeled by the $n$-fold product of $K_3$. Let us begin by defining the weak graph product of two graphs.

The weak product of $G$ and $H$, denoted by $G \times H$ is defined as follows: the vertex set of $G \times H$ is the Cartesian product of the vertex sets of $G$ and $H$. Two vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent in $G \times H$ if $g_1 g_2$ is an edge of $G$ and $h_1 h_2$ is an edge of $H$. The “times” symbol, $\times$, is supposed to be reminiscent of the weak product of two edges: $| \times^2 | = \times$. In this paper “graph product” will always mean the weak product.

In the first part of the paper we consider the interesting special case of the product of complete graphs on $r > 2$ vertices, $G = K^n_r = \times_{j=1}^n K_r$.

We then discuss a more general setting, considering other $r$-regular graphs as well.

When $G = K^n_r$, we identify the vertices of $G$ in the obvious way with the elements of $\mathbb{Z}^n_r$. Recalling the definition of the product, two vertices are adjacent in $G$ iff the corresponding vectors differ in every coordinate. Clearly one can color $G$ with $r$ colors by choosing a coordinate $i$ and coloring every vertex according to its $i$th coordinate. The following theorem asserts that if $r > 2$ then these are the only $r$-colorings. Here, and in what follows, we denote by $|H|$ the number of vertices of a graph $H$.

**Theorem 1.1.** Let $G = K^n_r$, and assume $r \geq 3$. Let $I$ be an independent set with $|I| = \frac{|G|}{r}$. Then there exists a coordinate $i \in \{1, \ldots, n\}$ and $k \in \{0, \ldots, r-1\}$ such that

$$I = \{v : v_i = k\}.$$

Consequently, the only colorings of $G$ by $r$ colors are those induced by colorings of one of the factors $K_r$.

Greenwell and Lovász [GL] proved the above theorem (and actually, a somewhat stronger statement) more than a quarter of a century ago. See also [Mii] for a similar result. The novelty in this paper is the proof we supply that uses Fourier analysis on the group $\mathbb{Z}^n_r$. Our approach also allows us to deduce a stability version of the above theorem.

**Theorem 1.2.** For every $r \geq 3$ there exists a constant $M = M(r)$ such that for any $\epsilon > 0$ the following is true. Let $G = K^n_r$. Let $J$ be an independent set such that $\frac{|J|}{|G|} = \frac{1}{r} - \epsilon$. Then there exists an independent set $I$ with $\frac{|I|}{|G|} = \frac{1}{r}$ such that $\frac{|J \Delta I|}{|G|} < M \epsilon$. 