RIGIDITY OF SPHERICAL BUILDINGS AND JOINS

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Abstract. We prove a rigidity and a characterization result for buildings and spherical joins using sets of antipodal points.

1 Introduction

1.1 Results and Motivation The goal of this paper is the following:

Main Theorem. Let $X$ be a finite dimensional geodesically complete CAT(1) space. If $X$ has a proper closed subset $A$ containing with each $x \in A$ all antipodes of $x$, i.e. all points $z \in X$ with $d(z,x) \geq \pi$, then $X$ is a spherical join or a building.

This theorem implies the following result announced in the title:

Corollary 1.1. Let $X$ be a non-discrete spherical building or a spherical join. If $f : X \rightarrow Y$ is a surjective 1-Lipschitz map onto a finite-dimensional geodesically complete CAT(1) space, then $Y$ is a spherical building or a spherical join too.

The above theorem was mainly motivated by an attempt to better understand the following deep rigidity result of Leeb [Le]:

Theorem. Let $H$ be a geodesically complete locally compact Hadamard space. If the ideal boundary $X$ of $H$ equipped with the Tits metric is a non-discrete irreducible spherical building, then $H$ is an affine building or a symmetric space.

The connection between our result and the theorem of Leeb is provided by the observation, that for each point $x$ in a Hadamard space $H$ as above, there is a natural (logarithmic) surjective 1-Lipschitz map $f : X \rightarrow S_x H$ from the ideal boundary $X$ onto the link at $x$. Our Corollary 1.1 implies at once that under the conditions of the theorem of Leeb each link of $H$ is itself a spherical join or a building. We hope that our methods will provide a generalization and another proof of the theorem of Leeb.

Another more direct motivation comes from the following theorem of Eberlein [E, p.340], that can be used to simplify the proof of the higher rank rigidity established in [B] and [BurS] (see [E] for a detailed exposition):
Theorem. Let $H$ be a Hadamard manifold, $X$ its ideal boundary. If $X$ contains a subset $A$, closed in the cone topology and involutive, then the holonomy group of $H$ is not transitive.

Involutive in the above theorem has essentially the same meaning as the condition in the Main Theorem. The result of Eberlein and the theorem of Berger and Simon imply that $H$ is a product or a symmetric space and therefore $X$ is a building or a spherical join with respect to the Tits metric [E].

One can get more precise statements about surjective Lipschitz maps as far as buildings or spherical joins are concerned. The spherical joins are much easier to understand, namely:

Corollary 1.2. Let $X$ be a finite dimensional geodesically complete CAT(1) space. Then $X$ has a unique decomposition $X = S^l \ast G_1 \ast \ldots \ast G_k \ast X_1 \ast \ldots \ast X_m$ where $G_j$ are thick irreducible buildings and $X_j$ are irreducible (i.e. indecomposable as a spherical join) but not buildings.

If $f : X_1 \ast X_2 \to Y$ is a surjective 1-Lipschitz map, where $X_1, X_2$ and $Y$ are CAT(1) spaces and $Y$ is geodesically complete, then $Y$ splits as $Y = f(X_1) \ast f(X_2)$. Moreover, $f$ is uniquely determined by the restrictions of $f$ to $X_1$ and to $X_2$.

On the side of buildings we first have a universal construction:

Corollary 1.3. For each finite dimensional geodesically complete CAT(1) space $X$, there is a building $\hat{X}$ with $\dim(\hat{X}) \leq \dim X$ and a bijective 1-Lipschitz map $i_X : \hat{X} \to X$ universal in the following sense: For each surjective 1-Lipschitz map $f : X \to Y$ to another finite dimensional geodesically complete CAT(1) space $Y$, the map $\hat{f} : \hat{X} \to \hat{Y}$ given by $\hat{f} = (i_Y)^{-1} \circ f \circ i_X$ is still surjective and 1-Lipschitz.

The building $\hat{X}$ arising from $X$ in a functorial way is in fact not really exotic. It is uniquely determined by the following properties: $i_X : \hat{X} \to X$ is an isometry iff $X$ is a building. $\hat{X}$ is discrete iff $X$ is neither a spherical join nor a building of dimension at least 1. Finally the functor $X \to \hat{X}$ respects spherical joins.

To complete the picture we have to describe how complicated surjective 1-Lipschitz maps between buildings can be. There are three basic types of such maps, corresponding to (very coarse) different types of Hadamard spaces.

1. If $G$ is discrete, then each surjective map $f : G \to X$ is 1-Lipschitz.

This describes the fact, that arbitrary CAT(1) spaces can occur in links negatively curved Hadamard spaces.