ANDERSON LOCALIZATION FOR QUASI-PERIODIC LATTICE SCHRÖDINGER OPERATORS ON $\mathbb{Z}^d$, $d$ ARBITRARY

JEAN BOURGAIN

0 Introduction

Consider the quasi-periodic lattice Schrödinger operator on $\mathbb{Z}^d$

$$\mathcal{H} = \lambda v(x + n\omega)\delta_{nn'} + \Delta \quad (0.1)$$

where $v$ is a real analytic potential on $\mathbb{T}^d$, $n\omega = (n_1\omega_1, \ldots, n_d\omega_d)$ and $\Delta$ stands for the lattice Laplacian

$$\Delta(n, n') = \begin{cases} 1 & \text{if } \sum |n_i - n'_i| = 1, \\ 0 & \text{otherwise}. \end{cases}$$

As in [BGS] (where quasi-periodic operators on $\mathbb{Z}^2$ are considered) we assume moreover that $v$ satisfies the following non-degeneracy condition (0.2):

(0.2) For all $i = 1, \ldots, d$ and $(\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_d) \in \mathbb{T}^{d-1}$, the map

$$\theta_i \mapsto v(\theta_1, \ldots, \theta_i, \ldots, \theta_d)$$

is a nonconstant function of $\theta_i \in \mathbb{T}$.

In this paper we do obtain the following extension of the [BGS] result ($d = 2$) to arbitrary dimension.

**Theorem.** Fix $x \in \mathbb{T}^d$, say $x_0 = 0$. Let $\delta > 0$.

Under the above assumption there is $\lambda(v, \delta)$ such that for $\lambda > \lambda(v, \delta)$, there is a subset $\Omega = \Omega_{\lambda v} \subset \mathbb{T}^d$ with

$$\text{mes}(\mathbb{T}^d \setminus \Omega) < \delta$$

and for $\omega \in \Omega$, $\mathcal{H}$ satisfies Anderson (and dynamical) localization.

**Keywords and phrases:** Anderson localization, semi-algebraic sets, Green’s functions

**AMS Mathematics Subject Classification:** 81Q10 (47B80, 47N50, 60J45, 82B44)

Supported by NSF grant 0627882.
Recall that Anderson localization means pure point spectrum with exponentially decaying eigenstates. Dynamical localization relates to the associated Schrödinger group $e^{itH}$, $t \in \mathbb{R}$, and roughly states that the flow is not diffusive. More precisely
\[
\sup_t \left[ \sum_{n \in \mathbb{Z}^d} |n|^2 \left| (e^{itH} \psi)(n) \right|^2 \right]^{1/2} < \infty
\]
whenever $\psi \in \ell^2(\mathbb{Z}^d)$ and with rapid decay at infinity (say $|\psi_n| < |n|^{-A}$ for $|n| \to \infty$ and $A$ large enough). This property implies in particular the absence of continuous spectrum.

This result is perturbative for $d > 1$ (contrary to the $d = 1$ case where we may take $\lambda > \lambda(v)$, see [BG]), cf. [B2].

The case $d = 2$ was treated in [BGS].

As in [BGS], there are two basic steps in the proof:

(I) Estimations on the Green’s function for fixed energy;

(II) Elimination of the energy.

In [BGS], step (I) was performed under an arithmetic condition on $\omega$, independent of the potential (which at least implies the absence of absolutely continuous spectrum). This condition roughly guarantees that if we consider an arbitrary algebraic curve $\Gamma$ in $[0,1]^2$ of ‘low’ degree, then the set of lattice points
\[
\{(k_1, k_2) \in \mathbb{Z}^2 \mid |k_i| < N \text{ and } (k_1 \omega_1, k_2 \omega_2) \text{ ‘very close’ to } \Gamma\}
\]
where $k_i \omega_i$ stands for the fractional part $\{k_i \omega_i\} \in [0,1]$, may be estimated by $N^{1-\delta}$, for some $\delta > 0$.

We do not know of a similar result for $d \geq 3$ when $\Gamma$ is an arbitrary algebraic hypersurface. This issue turns out to be the only obstruction to generalize [BGS] to higher dimension.

The way we resolve the problem here is by imposing already in step (I) a condition on $\omega$ that depends on the potential.

As in [BGS] this set of frequencies is then further restricted when performing step (II). (Also in [BGS], this second restriction is dependent on the potential.)

The basic techniques involved here are the same as in [BGS]; they are the theory of semi-algebraic sets and estimates on subharmonic functions. More specifically, we rely again on the matrix-valued Cartan-type theorem from [BGS] (see also [B2]) for certain real analytic operator-valued functions. It will be recalled in the Appendix for the convenience of the reader.

We believe that elimination methods similar to those used here deserve to be explored more since they may lead to improvements on the currently