COLLAPSING CONSTRUCTION WITH NILPOTENT STRUCTURES

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Abstract. A fundamental result concerning collapsed manifolds with bounded sectional curvature is the existence of compatible local nilpotent symmetry structures whose orbits capture all collapsed directions of the local geometry [CFG]. The underlying topological structure is called an $N$-structure of positive rank. We show that if a manifold $M$ admits such an $N$-structure $\mathcal{N}$, then $M$ admits a one-parameter family of metrics $g_\epsilon$ with curvature bounded in absolute value while injectivity radii and the diameters of $\mathcal{N}$-orbits away from the singular set of $\mathcal{N}$ uniformly converge to zero as $\epsilon \to 0$. Moreover, $g_\epsilon$ is $\mathcal{N}$-invariant away from the singular set. This result extends collapsing results in [CG1], [Fu3] and [G].

0 Introduction

Let $M$ be a Riemannian manifold. For $\epsilon > 0$, we call $M$ $\epsilon$-collapsed with bounded curvature, if the injectivity radii are everywhere smaller than $\epsilon$ while the sectional curvature is bounded in absolute value, say $|\text{sec}_M| \leq 1$. Collapsed manifolds with bounded curvature were extensively studied by Cheeger–Gromov and Fukaya ([CG1,2], [CFG], [Fu1-4], [G]), and the geometric and topological structures of the collapsed manifolds are understood well (see below). For recent applications, see [FR1,2], [PT], [PRT], and references within [R2].

Gromov’s theorem on almost flat manifolds ([G], [Ru]) characterizes a maximally collapsed $n$-manifold $M$, i.e. the diameter of $M$ is less than $\epsilon(n)$ (a constant depending on $n$) while $|\text{sec}_M| \leq 1$. It asserts that up to a small
perturbation of the metric, $M = \Gamma \backslash N$ is an infranilmanifold, where $N$ is a simply connected nilpotent Lie group equipped with a left-invariant metric and $\Gamma$ is a discrete subgroup of $N \rtimes \text{Aut}(N)$ such that $[\Gamma, \Gamma \cap N] \leq w(n)$, a constant depending on $n$.

For a general collapsed manifold $M$, its local structure can be described as follows (a detailed definition is given in section 1): each $x \in M$ has an open neighborhood $U$ (called a chart) that fibers over an infranilmanifold, $\Gamma \backslash N$, with fiber a Euclidean ball [CFG]. We call the cross section through $x$ an orbit. These charts fit together compatibly, i.e. when charts meet, the orbit from one chart sits in the orbit from the other chart. The compatibility implies that $M$ decomposes into orbits, and an orbit at $x$ is the union of orbits through $x$ of all possible charts containing $x$. Moreover, up to a small perturbation of the metric, an orbit, equipped with the induced metric, is an embedded infranilmanifold. This structure is called a nilpotent Killing structure on $M$ with respect to the perturbed metric. The smallest dimension of orbits of a nilpotent Killing structure $N$ is called the rank of $N$. An orbit is singular if any of its neighborhoods contains an orbit of strictly larger dimension. The singular set $S_N$ of $N$ is the union of all singular orbits. The center part of $\Gamma \backslash N$ forms a substructure of $N$, called its canonical $F$-structure, whose orbits are flat manifolds (see section 1).

The following is a fundamental result on collapsed manifolds with bounded sectional curvature.

**Theorem 0.1** [CFG]. There is a constant $\epsilon(n) > 0$ such that if the sectional curvature and injectivity radii of a complete Riemannian $n$-manifold $M$ satisfy

\[ |\text{sec}_M| \leq 1, \quad \text{Injrad}(M, x) < \epsilon \leq \epsilon(n), \quad \forall x \in M, \]

then $M$ admits a nilpotent Killing structure $\mathcal{N}$ with respect to a metric $g_\epsilon$ such that

(0.1.1) $|g - g_\epsilon|_{C^1} < \epsilon$, $|\text{sec}_{g_\epsilon}| \leq 1$ and $|\nabla^k R_{g_\epsilon}| \leq C(n, k, \epsilon)$, a constant depending on $n, k$ and $\epsilon$.

(0.1.2) Each $\mathcal{N}$-orbit has positive dimension and diameter less than $\epsilon$.

We call the underlying topological structure of a nilpotent Killing structure a nilpotent structure (simply, an $N$-structure) of positive rank. A natural question is whether such an $N$-structure on a manifold will imply a (arbitrarily) collapsed metric?

It was suggested in [CFG, p. 328] that the answer is positive, however it has only been known in the following cases: