CONSTANT SCALAR CURVATURE METRICS ON TORIC SURFACES

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Abstract. The main result of the paper is an existence theorem for a constant scalar curvature Kahler metric on a toric surface, assuming the K-stability of the manifold. The proof builds on earlier papers by the author, which reduce the problem to certain a priori estimates. These estimates are obtained using a combination of arguments from Riemannian geometry and convex analysis. The last part of the paper contains a discussion of the phenomena that can be expected when the K-stability does not hold and solutions do not exist.

1 Introduction

This paper continues the series [D1,2,3] in which we study the scalar curvature of Kahler metrics on toric varieties and relations with the analysis of convex functions on polytopes in Euclidean space. The main result of the present paper is an existence theorem for metrics of constant scalar curvature on toric surfaces, confirming a conjecture in [D1].

We begin by recalling the background briefly: more details can be found in the references above. Let $P$ be a bounded open polytope in $\mathbb{R}^n$ and let $\sigma$ be a measure on the boundary of $P$ which is a multiple of the standard Lebesgue measure on each face. Let $A$ be a smooth function on the closure $\overline{P}$ of $P$ and consider the linear functional $L_{A,\sigma}$ on the continuous functions on $\overline{P}$ given by

$$L_{A,\sigma}f = \int_{\partial P} f \, d\sigma - \int_P A f \, d\mu ,$$

where $d\mu$ is ordinary Lebesgue measure on $\mathbb{R}^n$. We define a nonlinear

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functional on a suitable class of convex functions \( u \) on \( P \) by
\[
\mathcal{M}(u) = -\int_P \log \det(u_{ij}) d\mu + L_{A,\sigma} u,
\]
where \((u_{ij})\) denotes the Hessian matrix of second derivatives of \( u \).

The Euler–Lagrange equation associated to \( \mathcal{M} \) is the fourth order PDE found by Abreu:
\[
\sum_{i,j} \frac{\partial^2 u_{ij}}{\partial x_i \partial x_j} = -A, \tag{1}
\]
where \((u_{ij})\) is the matrix inverse of \((u_{ij})\). (We often use the notation \( u_{ij}^{ij} \) for the left-hand side of (1).) A solution of this equation, with appropriate boundary behaviour, is a critical point, in fact the minimiser, of \( \mathcal{M} \). More precisely, we require \( u \) to satisfy Guillemin boundary conditions, which depend on the measure \( \sigma \). We refer to the previous papers cited for the details of these boundary conditions and just recall here the standard model for the behaviour at the boundary. This is the function
\[
u(x_1, \ldots, x_n) = \sum x_i \log x_i,
\]
on the convex subset \( \{x_i > 0\} \) in \( \mathbb{R}^n \). For appropriate “Delzant” pairs \((P, \sigma)\) a function \( u \) with this boundary behaviour defines a Kahler metric on a corresponding toric manifold \( X_P \). This comes with a map \( \pi : X_P \to P \) and the scalar curvature of the metric is \( A \circ \pi \). Thus when \( A \) is constant the Kahler metric has constant scalar curvature. When \( A \) is an affine-linear function the Kahler metric is “extremal”.

The Guillemin boundary conditions imply that
\[
L_{A,\sigma}(f) = \int_P \sum f_{ij} u_{ij} d\mu, \tag{2}
\]
for smooth test functions \( f \). Thus \( L_A \) vanishes on affine-linear functions. This just says that \((P, Ad\mu)\) and \((\partial P, d\sigma)\) have the same mass and moments. More interestingly, equation (2) tells us that \( L_{A,\sigma}(f) \geq 0 \) for smooth convex functions \( f \), with strict equality if \( f \) is not affine linear. This can be extended to more general convex functions \( f \). In [D1] we conjectured that this necessary condition, for the existence of a solution \( u \), is also sufficient. The present paper completes the proof of this in the case when the dimension \( n \) is 2 and the function \( A \) is a constant. Thus we have

**Theorem 1.** Suppose \( P \subset \mathbb{R}^2 \) is a polygon and \( \sigma \) is a measure on \( \partial P \), as above, with the property that the mass and moments of \((\partial P, \sigma)\) and \((P, Ad\mu)\) are equal for some constant \( A \). Then either there is a solution to (1), satisfying Guillemin boundary conditions, or there is a convex function \( f \), not affine linear, with \( L_{A,\sigma}(f) \leq 0 \).