SUBSHIFTS WITH SLOW COMPLEXITY AND SIMPLE GROUPS WITH THE LIOUVILLE PROPERTY

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Abstract. We study random walk on topological full groups of subshifts, and show the existence of infinite, finitely generated, simple groups with the Liouville property. Results by Matui and Juschenko-Monod have shown that the derived subgroups of topological full groups of minimal subshifts provide the first examples of finitely generated, simple amenable groups. We show that if the (not necessarily minimal) subshift has a complexity function that grows slowly enough (e.g. linearly), then every symmetric and finitely supported probability measure on the topological full group has trivial Poisson–Furstenberg boundary. We also get explicit upper bounds for the growth of Følner sets.

1 Introduction

In the early 50s Graham Higman [Hig51] gave the first example of a finitely generated, infinite simple group. Later, Hall [Hal74], Gorjuškin [Gor74], and Schupp [Sch76] showed that any countable group can be embedded in a 2-generated simple group. Thus, finitely generated simple groups can be arbitrarily “large” in some sense. It is considerably less understood how “small” can such groups be, from the point of view of their asymptotic geometry.

It follows from Gromov’s theorem [Gro81] that a finitely generated simple group can not have polynomial growth, and it is an open question, due to Grigorchuk (see [Gri13, Problem 15]), whether it can have sub-exponential growth. Recall that groups of sub-exponential growth are amenable. Recently, Juschenko and Monod [JM13] have proven that there do exist finitely generated, simple groups that are amenable; the groups that they consider were known to be simple and finitely generated by results of Matui [Mat06].

We consider here a third property of groups that lies between sub-exponential growth and amenability: the Liouville property for finite-range symmetric random walks. We prove that there exist simple groups with the Liouville property.

A group equipped with a probability measure \((G, \mu)\) has the Liouville property if the Poisson–Furstenberg boundary is trivial; equivalently, if every bounded \(\mu\)-harmonic function on \(G\) is constant on the subgroup generated by the support of \(\mu\). Here a function \(f : G \rightarrow \mathbb{R}\) is said to be \(\mu\)-harmonic if \(f * \mu = f\), where
\[(f * \mu)(g) = \sum_{h \in G} f(gh)\mu(h).\] If the support of \(\mu\) generates \(G\) the measure is said to be non-degenerate.

When no measure is specified, we say that the group \(G\) has the Liouville property if \((G, \mu)\) has the Liouville property for every symmetric, finitely supported probability measure \(\mu\) on \(G\), including degenerate measures. Finitely generated groups with sub-exponential growth have the Liouville property (this is due to Avez [Ave74]), and groups with the Liouville property are amenable. More precisely, a group is amenable if, and only if, it admits a symmetric non-degenerate measure \(\mu\) with trivial Poisson–Furstenberg boundary (one implication is due to Furstenberg, see [KV83, Theorem 4.2], the other to Kaimanovich and Vershik [KV83, Theorem 4.4] and to Rosenblatt [Ros81]). However, there are finitely generated amenable groups, such as the wreath product \(\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^3\), that admit no finitely supported, non-degenerate measures with trivial boundary, see Kaimanovich and Vershik [KV83, Proposition 6.1]; on some amenable groups, a non-degenerate measure with trivial boundary might not even be chosen to have finite entropy by a result of Erschler [Ers04, Theorem 3.1]. For a recent survey on Poisson–Furstenberg boundaries of random walks on discrete groups, see [Ers10].

**Theorem 1.1.** There exist finitely generated infinite groups that are simple and have the Liouville property (for every symmetric, finitely supported probability measure). Moreover, there are uncountably many pairwise non-isomorphic such groups.

Theorem 1.1 is a consequence of Theorem 1.2 below.

The groups that we consider to prove Theorem 1.1 are a sub-class of the finitely generated simple groups discovered by Matui [Mat06, Theorem 4.9, Theorem 5.4] and considered by Juschenko and Monod [JM13]. They arise as the commutator sub-groups of the topological full groups of some minimal subshifts, on which we assume that the complexity grows slowly enough (these notions are defined in Section 1.1). Our approach does not rely on results in [JM13] and yields a new proof of amenability of the groups that we consider. It also shows amenability of the topological full groups of a class of non-minimal subshifts with slow complexity (see Section 1.2).

### 1.1 Cantor systems, subshifts, and topological full groups

Throughout the paper let \(\Sigma\) denote a compact, metrizable, totally disconnected topological space, and let \(\tau\) be a homeomorphism of \(\Sigma\).

The topological full group of the dynamical system \((\Sigma, \tau)\) is the group \([[\tau]]\) of homeomorphisms of \(\Sigma\) that locally coincide with a power of \(\tau\), namely the group of homeomorphisms \(g \in \text{Homeo}(\Sigma)\) such that for every \(x \in \Sigma\) there exists an open neighbourhood \(U\) of \(x\) and an integer \(k \in \mathbb{Z}\) for which \(g|_U = \tau^k|_U\). In other words, \(g \in \text{Homeo}(\Sigma)\) belongs to \([[\tau]]\) if and only if there exists a continuous function \(k_g : \Sigma \rightarrow \mathbb{Z}\), the orbit cocycle, such that

\[\forall x \in \Sigma, \quad g(x) = \tau^{k_g(x)}(x).\]