NON-VEECH SURFACES IN $\mathcal{H}^{hyp}(4)$ ARE GENERIC

DUC-MANH NGUYEN AND ALEX WRIGHT

Abstract. We show that every surface in the component $\mathcal{H}^{hyp}(4)$, that is the moduli space of pairs $(M, \omega)$ where $M$ is a genus three hyperelliptic Riemann surface and $\omega$ is an Abelian differential having a single zero on $M$, is either a Veech surface or a generic surface, i.e. its $\text{GL}^+(2, \mathbb{R})$-orbit is either a closed or a dense subset of $\mathcal{H}^{hyp}(4)$. The proof develops new techniques applicable in general to the problem of classifying orbit closures, especially in low genus. Combined with work of Matheus and the second author, a corollary is that there are at most finitely many non-arithmetic Teichmüller curves (closed orbits of surfaces not covering the torus) in $\mathcal{H}^{hyp}(4)$.

1 Introduction

1.1 General motivation. There is a natural $\text{GL}^+(2, \mathbb{R})$ action on moduli spaces of translation surfaces, discussed in more detail below. Its study is the central purpose of Teichmüller dynamics, and has diverse applications to the dynamics and geometry of interval exchange transformations and rational billiards, and related problems in physics.

Typical questions about translation surfaces include: What are the asymptotics for the number of cylinders of length at most $L$? How can the surface be decomposed into simpler pieces, respecting the flat geometry? What is the dynamical behavior of the directional flow in most directions? What are the deviations of ergodic averages for the directional flow?

Precisely answering most of these questions for a given surface usually requires first knowing its orbit closure. Indeed, the orbit closure is the arena in which much of the study of a translation surface occurs: The $\text{GL}^+(2, \mathbb{R})$ action on this orbit closure provides the renormalization dynamics for the directional flow on the surface, and how the orbit closure sits inside of the ambient moduli space determines whether and how often certain geometric configurations can be found inside the surface.

1.2 Background. Let $\underline{k} = (k_1, \ldots, k_n)$, with $k_i \in \mathbb{N}$. Recall that $\mathcal{H}(\underline{k})$ is the moduli space of pairs $(M, \omega)$, where $M$ is a Riemann surface and $\omega$ is a holomorphic one-form (Abelian differential) on $M$ having $n$ zeros of orders $(k_1, \ldots, k_n)$. Elements
of $\mathcal{H}(k)$ are called translation surfaces. It is well known that $\mathcal{H}(k)$ is an algebraic variety and also a complex orbifold of dimension $2g+n-1$ (see [KZ03,MT02,Zor06]).

There is a natural action of $\text{GL}^+(2,\mathbb{R})$ on each stratum $\mathcal{H}(k)$. Every translation surface can be obtained from a collection of polygons in $\mathbb{R}^2$ by gluing pairs of parallel edges with equal length, and the $\text{GL}^+(2,\mathbb{R})$ action is obtained from the linear action on the polygons in $\mathbb{R}^2$. See the surveys [MT02,Zor06] for a more detailed introduction. It turns out that the geometric and dynamic features of a specific translation surface are usually encoded in the closure of its $\text{GL}^+(2,\mathbb{R})$-orbit (see [Zor06]).

The space $\mathcal{H}(k)$ carries a natural volume form called the Masur-Veech measure. By the work of Masur and Veech, the action of $\text{SL}(2,\mathbb{R})$ is ergodic on the locus of unit area surfaces in each connected component of $\mathcal{H}(k)$ with respect to this measure. As a consequence, for almost every surface $(M,\omega)$ in $\mathcal{H}(k)$, the $\text{GL}^+(2,\mathbb{R})$-orbit of $(M,\omega)$ is dense in a connected component of $\mathcal{H}(k)$. We will call such $(M,\omega)$ generic surfaces. On the other hand, every stratum contains infinitely many square-tiled surfaces, and these surfaces have closed $\text{GL}^+(2,\mathbb{R})$-orbits.

In genus two, a complete classification of $\text{GL}^+(2,\mathbb{R})$-orbit closures has been obtained by McMullen [McM07,McM05]. There are also some partial results by Calta [Cal04] and Hubert-Lelièvre [HL06]. Using similar ideas, explicit examples of generic surfaces in the hyperelliptic locus $\mathcal{L} \subset \mathcal{H}(2)$ and in $\mathcal{H}^{\text{hyp}}(4)$ were constructed by Hubert-Lanneau-Møller and the first author [HLM09,HLM12,Ngu11].

1.3 Recent progress and hopes for the future. Recently Eskin–Mirzakhani [EM13], and Eskin–Mirzakhani–Mohammadi [EMM13], with a contribution by Avila–Eskin–Møller [AEM12], proved that all $\text{GL}^+(2,\mathbb{R})$-orbit closures are submanifolds of $\mathcal{H}(k)$ which are complex linear subspaces in local period coordinates (see Section 2). This confirmed a longstanding conjecture and has lead to further progress. The structure theory of affine invariant manifolds was developed by the second author in [Wri12], in particular leading to an explicit full measure set of generic translation surfaces in each stratum. Furthermore the geometry of translation surfaces has been directly connected to orbit closures via the Cylinder Deformation Theorem of the second author in [Wri13a]; this will be one of our main tools below.

Conjectures of Mirzakhani (see [Wri12]) predict that there are in fact very few orbit closures, and those that do exist have special properties enjoyed by the current list of known examples. This work is the first partial verification of these conjectures in genus greater than 2.

1.4 Statement of result. Kontsevich-Zorich have classified the connected components of strata [KZ03]. In particular, there are always at most three connected components. The stratum $\mathcal{H}(4)$ has only two components: $\mathcal{H}^{\text{hyp}}(4)$ and $\mathcal{H}^{\text{odd}}(4)$. Here $\mathcal{H}^{\text{hyp}}(4)$ is the space of pairs $(M,\omega) \in \mathcal{H}(4)$ where $M$ is a hyperelliptic Riemann surface, and $\mathcal{H}^{\text{odd}}(4)$ consists of pairs $(M,\omega)$ where $\omega$ defines an odd spin structure on $M$ (see [KZ03] for a more detailed explanation).