RELATIVE VOLUME COMPARISON WITH INTEGRAL CURVATURE BOUNDS

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Abstract
In this paper we shall generalize the Bishop-Gromov relative volume comparison estimate to a situation where one only has an integral bound for the part of the Ricci curvature which lies below a given number. This will yield several compactness and pinching theorems.

1 Introduction
In this paper we shall be concerned with proving some results related to the work from [PeSW]. The techniques here are similar, but independent of the developments in [PeSW], still we suggest that the reader read at least the introduction to [PeSW]. Also our techniques are different from those used in [G] and [Y]. This paper can therefore be read without prior knowledge of those papers. We will present a new relative volume comparison estimate which generalizes the classical Bishop-Gromov comparison inequality. The consequences of this are manifold and hopefully far reaching.

To state our results we need some notation. On a Riemannian manifold $M$ define the function $g : M \to [0, \infty)$ as $g(x) =$ the smallest eigenvalue for $\text{Ric} : T_xM \to T_xM$. Now consider

$$k(\lambda, p) = \int_M \left( \max \{-g(x) + (n-1) \cdot \lambda, 0\} \right)^p \text{vol}$$

$$\bar{k}(\lambda, p) = \frac{1}{\text{vol} M} \int_M \left( \max \{-g(x) + (n-1) \cdot \lambda, 0\} \right)^p \text{vol},$$

the last quantity is the averaged amount of curvature below $(n-1) \lambda$. This quantity is in many ways more natural than the first. We of course have that $\text{Ric}(M) \geq (n-1) \lambda$ iff $\bar{k}(\lambda, p) = 0$.

Our main result is

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Theorem 1.1. Let $x \in M$, $\lambda \leq 0$, and $p > n/2$ be given, then there is a constant $C(n, p, \lambda, R)$ which is nondecreasing in $R$ such that when $r < R$ we have

$$
\left( \frac{\text{vol} B(x, R)}{v(n, \lambda, R)} \right)^{1/2p} - \left( \frac{\text{vol} B(x, r)}{v(n, \lambda, r)} \right)^{1/2p} \leq C(n, p, \lambda, R) \cdot (k(\lambda, p))^{1/2p}.
$$

Furthermore when $r = 0$ we obtain

$$\text{vol} B(x, R) \leq (1 + C(n, p, \lambda, R) \cdot (k(\lambda, p))^{1/2p})^2 v(n, \lambda, R).$$

Note that when $\text{Ric} \geq (n - 1) \lambda$, i.e. $k(p, \lambda) = 0$, this gives the classical relative volume comparison estimate. Our proof of this volume estimate does not use the setup used in Gallot and Yang's work. In fact our proof is somewhat different and is more inspired by some of the new estimates obtained in [PeSW]. The absolute volume estimate is actually better than the one in [Y], as we recover the correct volume estimates when $k(\lambda, p) = 0$. In [Y] one only arrives at the correct volume bound when $k(0, p) = 0$. As a corollary we obtain the following volume doubling result:

Corollary 1.2. Under the same conditions as the above theorem we have that for all $\alpha < 1$ there is an $\varepsilon = \varepsilon(n, p, \lambda, D, \alpha) > 0$ such that if $M$ is a Riemannian manifold with $\text{diam} M \leq D$ and $\tilde{k}(\lambda, p) \leq \varepsilon$, then for all $x \in M$ and $r < D$ we have

$$\alpha \cdot \frac{v(n, \lambda, r)}{v(n, \lambda, D)} \leq \frac{\text{vol} B(x, r)}{\text{vol} M}.$$

As an immediate consequence we have the following extension of Gromov’s precompactness result:

Corollary 1.3. Given an integer $n > 1$, $p > n/2$ and $\lambda \leq 0$, $D < \infty$, we can find $\varepsilon(n, p, \lambda, D)$ such that the class of closed Riemannian $n$-manifolds with

- $\text{diam} M \leq D$,
- $\tilde{k}(\lambda, p) \leq \varepsilon$

is precompact in the Gromov-Hausdorff topology.

The classical relative volume comparison result has proven very useful in many contexts. So one would expect the above inequality to give some new results that are similar but more general. Here we shall concentrate on compactness results. In a future paper we will show how other finiteness result also generalize when we use our new volume estimate.

The relative volume estimate will enable us to obtain some very general compactness and pinching results, where in addition to assuming lower