COMPLEX PROJECTIVE SURFACES AND INFINITE GROUPS

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1 Introduction

It is well known that complex projective surfaces can have highly nontrivial fundamental groups. It is also known that not all finitely presented groups can occur as fundamental groups of projective surfaces. The fundamental problem in the theory then is to determine which groups can occur as the fundamental groups of the complex projective surfaces and to describe complex manifolds which occur as nonramified coverings of the surface. The only interesting case is when the group is infinite since finite coverings of projective surfaces are projective and every finite group occurs as a fundamental group of some projective surface.

In this paper we mainly consider surfaces with a “representation” as a family of projective curves over a curve. This is quite general situation since any surface has such a representation after blowing up a finite number of points. We use a base change construction combined with a finite ramified covering to associate a collection of surfaces with infinite fundamental groups to a given surface. Most of these fundamental groups were not previously known to be fundamental groups of smooth projective surfaces. Every surface we construct comes equipped with a regular map to a curve of high genus. The kernel of the corresponding map of fundamental groups is obtained from the fundamental group of a generic fiber by imposing torsion relations on some elements and a very large class of infinite groups occur in this way. We also analyze the universal coverings of these surfaces in the context of the Shafarevich’s conjecture which states that the universal covering of the smooth complex projective variety must be holomorphically convex.

The first author is partially supported by DMS Grant-9500774 and the second author is partially supported by A.P. Sloan Dissertational Fellowship and by NSF Grant DMS 9700605.
We begin with a local version of the general construction. Namely we have a local fibration without multiple components and with only double singular points in the central fiber. As a first step we make a local base change which produces a new surface with singular points corresponding to the singular points of the central fiber. This move changes the image of the fundamental group of a generic fiber in the fundamental group of the Zariski open subset of nonsingular points. Namely the kernel of this map is generated by the $N$-th powers of the initial vanishing cycle where $N$ is the degree of the local base change. As a second step we desingularize the surface by taking a finite fiberwise covering of the singular surface which is ramified at singular points only.

The global construction follows the same pattern, but as a result we obtain a surface with a highly nontrivial fundamental group coming from the fiber even if the surface at the beginning was simply connected. First we construct a singular projective surface with a large fundamental group of the compliment to the set of singular points. A smooth projective surface is obtained at the second step as a finite covering of the singular surface ramified at singular points only. We prove a general result which establishes a close similarity between the fundamental groups and universal coverings for two classes of surfaces: normal projective surfaces and the surfaces obtained from them by deleting a finite number of points. The above construction enlarges the image of the fundamental group of the fiber in the fundamental group of the whole surface. The resulting group can be described in purely algebraic terms. Let $\pi_g$ be a fundamental group of a projective curve of genus $g$ (generic fiber of the fibration). Consider a finite set of pairs of $(s_i \in \pi_g, N_i \in \mathbb{N})$ and a subgroup $M$ of the automorphisms of $\pi_g$. We assume that all $s_i$ are vanishing cycles. These are special conjugacy classes in the fundamental group of the curve which constitute a finite number of orbits under the action of the mapping class group $Map(g)$ (see section 2). The orbits $M^N_i s_i$ generate the normal subgroup $\Xi(M, s_i, N_i)$ of $\pi_g$.

Now we can give an algebraic version of the description of the corresponding group.

**Definition 1.1.** Define a Burnside type quotient of $\pi_g$ to be the group $\pi_g/\Xi(M, s_i, N_i)$.

In the geometric situation $s_i$ are the vanishing cycles of the initial fibration and $M$ is its monodromy group. Geometric Burnside type groups constitute a proper subset among all Burnside type groups. In particular not every data $s_i, N_i, M$ can be geometrically realized. The problem which