UNIVERSAL COVERING MAPS AND RADIAL VARIATION

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1 Introduction and Statement of Results

We let $E \subseteq \mathbb{C}$ be a closed set with two or more points. By the uniformization theorem there exists a Fuchsian group of Moebius transformations such that $\mathbb{C} \setminus E$ is conformally equivalent to the quotient manifold $\mathbb{D}/G$. The universal covering map $P : \mathbb{D} \to \mathbb{C} \setminus E$ is then given by $P = \tau \circ \pi$, where $\pi$ is the natural quotient map onto $\mathbb{D}/G$ and $\tau$ is the conformal bijection between $\mathbb{C} \setminus E$ and $\mathbb{D}/G$. In this paper we will show that there exists $e^{i\beta} \in \mathbb{T}$ such that

$$\int_0^1 |P''(re^{i\beta})|dr < \infty. \quad (1.1)$$

Considering $u = \log |P'|$, one obtains (1.1) from variational estimates.

**Theorem 1.** There exists $e^{i\beta} \in \mathbb{T}$ and $M > 0$ such that for $r < 1$,

$$u(re^{i\beta}) < -\frac{1}{M} \int_0^r |\nabla u(re^{i\beta})|d\rho + M.$$ 

The class of universal covering maps contains two extremal cases. The case where $\mathbb{C} \setminus E$ is simply connected and the case where $E$ consists of two points. We considered the simply connected case in [JMii] where we proved that Anderson’s conjecture is true. The second case is easier: well known estimates for the Poincaré metric on the triply punctured sphere give (1.1) when $P$ is the universal covering of $\mathbb{C} \setminus \{0,1\}$.

In the course of the proof of Theorem 1 we measure the thickness of $E$ at all scales, and we are guided by the following philosophy. If, at some scale, the boundary $E$ appears to be thick then, locally, the universal covering map behaves like a Riemann map. On the other hand, if $E$ appears to be thin, then, locally, the Poincaré metric of $\mathbb{C} \setminus E$ behaves like the corresponding Poincaré metric of $\mathbb{C} \setminus \{0,1\}$. With the right estimates for

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the transition from the thick case to the thin case, this philosophy leads to a rigorous proof. Our method gives countably many good directions (these are the ones for which (1.1) holds). We point out that one cannot obtain a larger set of good directions in general. The simplest example showing this is the universal covering map onto \( \mathbb{C} \setminus \{0, 1\} \). Indeed this map fails to have radial limits, except for countably many rays. More refined examples are given in section 5. There we construct a domain such that the set of good rays is at most countable, and such that the radial limits of its universal covering map exist almost everywhere.

The following propositions present the main technical results of this paper. Each proposition gives estimates on the radial variation of \( u = \log |P'| \).

The hypothesis of Proposition 1 covers the case when to an observer at \( w = P(\zeta) \) the boundary \( E \) looks like a connected set. The hypothesis of Proposition 2 covers the case when the boundary \( E \) looks like an isolated point. To express these alternatives analytically, we use the function \( M(\zeta) = \sup_{z \in T(\zeta)} |\nabla u(z)| (1 - |z|) \)

where

\[ T(\zeta) = \left\{ w \in \mathbb{D} : |w - \zeta| \leq 1 - |\zeta|, (1 - |\zeta|)/2 \leq 1 - |w| \leq 1 - |\zeta| \right\} . \]

The first alternative corresponds to the case where \( u = \log |P'| \) satisfies a Bloch condition near \( \zeta \). The second alternative causes the failure of Bloch estimates near \( \zeta \). Correspondingly the proof of Proposition 1 uses the condition

\[ M(\zeta) \leq \text{some constant} , \]

whereas Proposition 2 requires that

\[ M(\zeta) \geq \text{a very large constant} . \]

Further combinatorial considerations provide the tools for an iterative solution of Theorem 1 based on repeated applications of Propositions 1 and 2.

In both Proposition 1 and 2 the following family of curves plays an important role. We let \( L \geq 1 \) be a positive integer, and we let \( z_1, z_2 \in \mathbb{D} \), \( |z_1| < |z_2| \). Then \( \Gamma(z_1, z_2, L) \) is the collection of all radial line segments

\[ \gamma = \{ s \in \mathbb{D} : |z_1| < |s| \leq |z_2| \} \cap (0, t) , \]

where \( t \in \mathbb{D} \) satisfies \( |t| = |z_2| \), \( |t - z_2| \leq 2^L(1 - |z_2|) \) and where \( (0, t) \) denotes the ray connecting \( 0 \in \mathbb{D} \) to \( t \in \mathbb{D} \).

We let \( M_1, L \) be positive integers and we fix a point \( \zeta \in \mathbb{D} \). Under the hypothesis that \( M(\zeta) < C \), the universal covering map \( P \) behaves locally like a Riemann map. Hence in the proof of Proposition 1 we work with