A semantical proof of the strong normalization theorem for full propositional classical natural deduction

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Abstract. We give in this paper a short semantical proof of the strong normalization for full propositional classical natural deduction. This proof is an adaptation of reducibility candidates introduced by J.-Y. Girard and simplified to the classical case by M. Parigot.

1. Introduction

This paper gives a semantical proof of the strong normalization of the cut-elimination procedure for full propositional classical logic written in natural deduction style. By full we mean that all the logical connectives (⊥, →, ∧ and ∨) are considered as primitive. We also consider the three reduction relations (logical, commutative and classical reductions) necessary to obtain the subformula property (see [5]).

Until very recently (see the introduction of [5] for a brief history), no proof of the strong normalization of the cut-elimination procedure was known for full logic.

In [5], Ph. De Groote gives such a proof by using a CPS-style transformation from full classical logic to implicative intuitionistic logic, i.e., the simply typed λ-calculus.

A very elegant and direct proof of the strong normalization of the full logic is given in [6] but only the intuitionistic case is given.

R. David and the first author give in [3] a direct and syntactical proof of this result. This proof is based on a characterization of the strongly normalizable deductions and a substitution lemma which stipulates the fact that the deduction obtained while replacing in a strongly normalizable deduction an hypothesis by another strongly normalizable deduction is also strongly normalizable. The same idea is used in [2] to give a short proof of the strong normalization of the simply typed λµ-calculus of [9].

R. Matthes recently found another semantical proof of this result (see [7]). His proof uses a complicated concept of saturated subsets of terms.

Our proof is a generalization of M. Parigot’s strong normalization result of the λµ-calculus (see [10]) for the types of J.-Y. Girard’s system $\mathcal{F}$ using reducibility candidates. We also use a very technical lemma proved in [3] concerning
commutative reductions. To the best of our knowledge, this is the shortest proof of a such result.

The paper is organized as follows. In section 2, we give the syntax of the terms and the reduction rules. In section 3, we define the reducibility candidates and establish some important properties. In section 4, we show an “adequation lemma” which allows to prove the strong normalization of all typed terms.

2. The typed system

We use notations inspired by the paper [1].

Definition 2.1 1. The types are built from propositional variables and the constant symbol \( \bot \) with the connectors \( \rightarrow, \land \) and \( \lor \).

2. Let \( \mathcal{X} \) and \( \mathcal{A} \) be two disjoint alphabets for distinguishing the \( \lambda \)-variables and \( \mu \)-variables respectively. We code deductions by using a set of terms \( T \) which extends the \( \lambda \)-terms and is given by the following grammars:

\[
T := \mathcal{X} | \lambda \mathcal{X}.T | (T \ E) | (T, T) \ | \omega_1 T \ | \omega_2 T \ | \mu \mathcal{A}.T \ | (A.T)
\]

\[
E := T \ | \pi_1 \ | \pi_2 \ | [X.T, \mathcal{X}.T]
\]

An element of the set \( E \) is said to be an \( E \)-term.

3. The meaning of the new constructors is given by the typing rules below where \( \Gamma \) (resp. \( \Delta \)) is a context, i.e. a set of declarations of the form \( x : A \) (resp. \( a : A \)) where \( x \) is a \( \lambda \)-variable (resp. \( a \) is a \( \mu \)-variable) and \( A \) is a type.

\[
\frac{\Gamma, x : A \vdash t : B; \Delta}{\Gamma \vdash \lambda x.t : A \rightarrow B; \Delta} \quad \frac{\Gamma \vdash u : A \rightarrow B; \Delta \quad \Gamma \vdash v : A; \Delta}{\Gamma \vdash (u \ v) : B; \Delta}
\]

\[
\frac{\Gamma \vdash u : A; \Delta \quad \Gamma \vdash v : B; \Delta}{\Gamma \vdash (u, v) : A \land B; \Delta} \quad \frac{\Gamma \vdash t : A \land B; \Delta}{\Gamma \vdash \langle t \ \pi_1 \rangle : A; \Delta} \quad \frac{\Gamma \vdash t : A \land B; \Delta}{\Gamma \vdash \langle t \ \pi_2 \rangle : B; \Delta}
\]

\[
\frac{\Gamma \vdash t : A; \Delta}{\Gamma \vdash \omega_1 t : A \lor B; \Delta} \quad \frac{\Gamma \vdash t : B; \Delta}{\Gamma \vdash \omega_2 t : A \lor B; \Delta}
\]

\[
\frac{\Gamma \vdash t : A \lor B; \Delta \quad \Gamma, x : A \vdash u : C; \Delta \quad \Gamma, y : B \vdash v : C; \Delta}{\Gamma \vdash (t \ [x.u, y.v]) : C; \Delta}
\]

\[
\frac{\Gamma \vdash t : A; \Delta \quad \Gamma \vdash a : A}{\Gamma \vdash (a \ t) : \bot; \Delta} \quad \frac{\Gamma \vdash t : \bot; \Delta \quad \Gamma \vdash a : A}{\Gamma \vdash \mu a.t : A; \Delta}
\]