Implicational (semilinear) logics I: a new hierarchy

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Abstract In abstract algebraic logic, the general study of propositional non-classical logics has been traditionally based on the abstraction of the Lindenbaum-Tarski process. In this process one considers the Leibniz relation of indiscernible formulae. Such approach has resulted in a classification of logics partly based on generalizations of equivalence connectives: the Leibniz hierarchy. This paper performs an analogous abstract study of non-classical logics based on the kind of generalized implication connectives they possess. It yields a new classification of logics expanding Leibniz hierarchy: the hierarchy of implicational logics. In this framework the notion of implicational semilinear logic can be naturally introduced as a property of the implication, namely a logic L is an implicational semilinear logic iff it has an implication such that L is complete w.r.t. the matrices where the implication induces a linear order, a property which is typically satisfied by well-known systems of fuzzy logic. The hierarchy of implicational logics is then restricted to the semilinear case obtaining a classification of implicational semilinear logics that encompasses almost all the known examples of fuzzy logics and suggests new directions for research in the field.

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1 Introduction

Algebraic logic is the branch of mathematical logic that studies logical systems by giving them a semantics based on some particular kind of algebraic structures. It can be traced back to George Boole and his study of classical propositional logic by means of a two-element algebra that became its canonical semantics. Thus, in a sense, it could be argued that algebraic logic is the oldest branch of mathematical logic.

Tarski’s refinement of the proof by Linbenbaum that classical logic is indeed complete with respect to the semantics given by Boolean algebras starts from a theory \( T \) and a formula \( \varphi \) such that \( T \not\vdash_{\text{CPC}} \varphi \), i.e. \( T \) does not prove \( \varphi \) in the classical propositional calculus, and then it considers the following binary relation on the set of formulae:

\[
\langle \alpha, \beta \rangle \in \Omega(T) \iff T \vdash_{\text{CPC}} \alpha \iff \beta.
\]

This relation is shown to be in fact a congruence in the algebra of formulae \( \text{Fm}_L \); moreover the formulae of \( T \) constitute exactly one equivalence class. Thus it is enough to take the corresponding quotient, \( \text{Fm}_L / \Omega(T) \), and show that it is a Boolean algebra such that the class of \( T \) is its top element, and hence in this algebra the elements of \( T \) are interpreted as true while \( \varphi \) is not (because \( T \not\vdash_{\text{CPC}} \varphi \)).

Analogous proofs were later used to show the completeness of non-classical logics with respect to their corresponding algebraic semantics (e.g. intuitionistic logic w.r.t. Heyting algebras); indeed, it became known in algebraic logic as the standard method called the Lindenbaum-Tarski process. The fact that it could be analogously repeated in many propositional logics led to more general studies where it was used to show completeness theorems for broad classes of logics such as Rasiowa’s implicative logics (studied in her monograph [24]). Abstract algebraic logic (AAL) was born as the natural next step to be taken in this evolution: the abstract study of logical systems through the generalization of the Lindenbaum-Tarski process to arbitrary logics. The last decades have seen the florescence of this subfield of algebraic logic resulting in a deep theory of the correspondence between logics and classes of algebras (or logical matrices defined over the algebras). The generalization of the Lindenbaum-Tarski construction capitalizes on the realization that the congruence \( \Omega(T) \) is actually the relation consisting of those pairs of formulae that, relatively to \( T \), are substitutable in any context salva veritate, i.e.:

\[
\langle \alpha, \beta \rangle \in \Omega(T) \text{ if, and only if, for every formula in at least one variable } \chi(x), \chi(\alpha) \text{ is true relatively to } T \text{ iff } \chi(\beta) \text{ is true relatively to } T.
\]