We study the relationship between fields of transseries and residue fields of convex subrings of non-standard extensions of the real numbers. This was motivated by a question of Todorov and Vernaeve, answered in this paper.

Keywords Transseries · Nonstandard extensions of the real field · O-Minimality

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In this note we answer a question by Todorov and Vernaeve (see, e.g., [25]) concerning the relationship between the field of logarithmic-exponential series from [35] and the residue field of a certain convex subring of a non-standard extension of the real numbers, introduced in [24] in connection with a non-standard approach to Colombeau’s theory of generalized functions. The answer to this question can almost immediately be deduced from well-known (but non-trivial) results about o-minimal structures. It should therefore certainly be familiar to logicians working in this area, but perhaps less so to those in non-standard analysis, and hence may be worth recording.

We begin by explaining the question of Todorov–Vernaeve. Let $^*\mathbb{R}$ be a non-standard extension of $\mathbb{R}$. Given $X \subseteq \mathbb{R}^m$ we denote the non-standard extension of $X$ by $^*X$. 

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and given also a map \( f : X \to \mathbb{R}^n \), by abuse of notation we denote the non-standard extension of \( f \) to a map \(*X \to *\mathbb{R}^n\) by the same symbol \( f \).

Let \( O \) be a convex subring of \(*\mathbb{R} \). Then \( O \) is a valuation ring of \(*\mathbb{R} \), with maximal ideal

\[
\mathfrak{m} := \{ x \in *\mathbb{R} : x = 0, \text{ or } x \neq 0 \text{ and } x^{-1} \notin O \}.
\]

We denote the residue field \( O/\mathfrak{m} \) of \( O \) by \( \hat{O} \), with natural surjective morphism

\[
x \mapsto \hat{x} := x + \mathfrak{m} : O \to \hat{O}.
\]

The ordering of \(*\mathbb{R} \) induces an ordering of \( \hat{O} \) making \( \hat{O} \) an ordered field and \( x \mapsto \hat{x} \) order-preserving. By standard facts from real algebra [15], \( \hat{O} \) is real closed. Residue fields of convex subrings of \(*\mathbb{R} \) are called “asymptotic fields” in [24] (although this terminology is already used with a different meaning elsewhere [2]). The collection of convex subrings of \(*\mathbb{R} \) is linearly ordered by inclusion, and the smallest convex subring of \(*\mathbb{R} \) is

\[
*\mathbb{R}_{\text{fin}} = \{ x \in *\mathbb{R} : |x| \leq n \text{ for some } n \},
\]

with maximal ideal

\[
*\mathbb{R}_{\text{inf}} = \{ x \in *\mathbb{R} : |x| \leq \frac{1}{n} \text{ for all } n > 0 \}.
\]

The inclusions \( \mathbb{R} \to *\mathbb{R}_{\text{fin}} \to O \) give rise to a field embedding \( \mathbb{R} \to \hat{O} \), by which we identify \( \mathbb{R} \) with a subfield of \( \hat{O} \). In the case \( O = *\mathbb{R}_{\text{fin}} \) we have \( \hat{O} = \mathbb{R} \), and \( \hat{x} \) is the standard part of \( x \in *\mathbb{R}_{\text{fin}} \), also denoted in the following by \( \text{st}(x) \).

Let now \( \xi \in *\mathbb{R} \) with \( \xi > \mathbb{R} \) and let \( E \) be the smallest convex subring of \(*\mathbb{R} \) containing all iterated exponentials \( \xi, \exp \xi, \exp \exp \xi, \ldots \) of \( \xi \), that is,

\[
E = \{ x \in *\mathbb{R} : |x| \leq \exp_n(\xi) \text{ for some } n \},
\]

where \( \exp_0(\xi) = \xi \) and \( \exp_n(\xi) = \exp(\exp_{n-1}(\xi)) \) for \( n > 0 \). The maximal ideal of \( E \) is

\[
\mathfrak{e} = \{ x \in *\mathbb{R} : |x| \leq \frac{1}{\exp_n(\xi)} \text{ for all } n \},
\]

with residue field \( \hat{E} = E/\mathfrak{e} \). Note that the definition of \( \hat{E} \) depends on the choice of \(*\mathbb{R} \) and \( \xi \), which is suppressed in our notation; in [24,25], \( \hat{E} \) is denoted by \( \hat{E}_\varrho \) where \( \varrho = 1/\xi \).

By an exponential field we mean an ordered field \( K \) equipped with an exponential function on \( K \), i.e., an isomorphism \( f \mapsto \exp(f) \) between the ordered additive group of \( K \) and the ordered multiplicative group \( K^{>0} \) of positive elements of \( K \). We often write \( e^f \) instead of \( \exp(f) \), and the inverse of \( \exp \) is denoted by log: \( K^{>0} \to K \). It is