On VC-minimal fields and dp-smallness

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Abstract In this paper, we show that VC-minimal ordered fields are real closed. We introduce a notion, strictly between convexly orderable and dp-minimal, that we call dp-small, and show that this is enough to characterize many algebraic theories. For example, dp-small ordered groups are abelian divisible and dp-small ordered fields are real closed.

Keywords Convexly orderable · VC-minimal · Ordered fields

Mathematics Subject Classification (2010) 03C60 · 03C45 · 12J15

0 Introduction

Recently, model theorists have been working on using the progress in stability theory as a template for work in unstable theories. Since much of modern mathematics is done outside the stable world, it seems reasonable to explore such avenues.

The notion of a good “minimality” condition comes up frequently in stability theory, and there have been many useful suggestions for a suitable “minimality” condition in the unstable context. S. Shelah developed dp-minimality, which was subsequently studied extensively by many others [4, 7, 12, 14]. Another property, strictly stronger than dp-minimality, that was extensively studied is weak o-minimality [11]. In [1], Adler...
introduces the notion of VC-minimality, which sits strictly between weak o-minimality and dp-minimality. This too has been studied a great deal recently [3,5,6,8]. In [8], this author and M. C. Laskowski develop a new “minimality” notion called “convex orderability,” which sits strictly between VC-minimality and dp-minimality.

When turned toward specific classes of theories, these minimality properties can yield strong classification results. For example, Theorem 5.1 of [11] asserts that every weakly o-minimal ordered group is abelian divisible and Theorem 5.3 of [11] says that every weakly o-minimal ordered field is real closed. For another example, Proposition 3.1 of [14] yields that every dp-minimal group is abelian by finite exponent and Proposition 3.3 of [14] states that every dp-minimal ordered group is abelian. In a similar spirit, Flenner and this author show, in [6], that all convexly orderable ordered groups are abelian divisible.

The goal of this paper is two-fold. In the first part of this paper, we introduce a new “minimality” condition that we call “dp-smallness,” which fits strictly between convex orderability and dp-minimality. We then show that most of the results of [6] work when we replace “convexly orderable” with “dp-small.” The second part of the paper is devoted to answering, in the affirmative, Open Question 3.7 of [6]. That is, we show that every convexly orderable ordered field is real closed. Stronger than that, we actually show this for dp-small ordered fields.

**Notation.** Throughout this paper, let $T$ be a complete theory in a language $L$ with monster model $\mathcal{U}$. We will use $x$, $y$, $z$, etc. to stand for tuples of variables (instead of the cumbersome $\mathcal{F}$ or $x$). For any $A \subseteq \mathcal{U}$ and tuple $x$, let $A_x$ denote the set of all $|x|$-tuples from $A$ (so $A_x = A^{|x|}$). If $|x| = 1$, we will say that $x$ is of the home sort.\(^1\)

For a formula $\varphi(x)$ and $A \subseteq \mathcal{U}$, let

$$\varphi(A) = \{ a \in A_x : \mathcal{U} \models \varphi(a) \}.$$ 

For ordered groups $G$, let $G_+$ denote the set of positive elements of $G$. Similarly define $F_+$ for ordered fields $F$. For a dense ordered group $G$, let $\overline{G}$ denote the completion of $G$ (in the sense of the ordering). For valued fields $(F, v, \Gamma)$ (where $v : F^\times \rightarrow \Gamma$ is the valuation), for $a, b \in F$, let $a \sim b$ hold if and only if $v(a) \leq v(b)$.

**Outline.** In Sect. 1, we give all the relevant definitions and state the main results of the paper. We define dp-smallness in Definition 1.4. Theorem 1.6 shows that many of the results of [6] hold for dp-smallness instead of convex orderability. Finally, Theorem 1.7 states that all dp-small ordered fields are real closed, generalizing Theorem 5.3 of [11]. In Sect. 2, we provide a proof that dp-smallness does fit strictly between convex orderability and dp-minimality. We also prove Theorem 1.6. In Sect. 3, we prove Theorem 1.7. In Sect. 4, we discuss VC-minimal fields in general. We show that VC-minimal stable fields are algebraically closed and conjecture that all VC-minimal fields are either algebraically closed or real closed.

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1 We could also consider theories with multiple sorts, but for the purposes of this paper, we will need a single “home sort.”