Admissibility and refutation: some characterisations of intermediate logics

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Abstract Refutation systems are formal systems for inferring the falsity of formulae. These systems can, in particular, be used to syntactically characterise logics. In this paper, we explore the close connection between refutation systems and admissible rules. We develop technical machinery to construct refutation systems, employing techniques from the study of admissible rules. Concretely, we provide a refutation system for the intermediate logics of bounded branching, known as the Gabbay–de Jongh logics. We show that this gives a characterisation of these logics in terms of their admissible rules. To illustrate the technique, we also provide a refutation system for Medvedev’s logic.

Keywords Intermediate logic · Admissible rules · Refutation · Gabbay–de Jongh logics · Medvedev’s logic

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According to Łukasiewicz [39], we assert true propositions, and reject false ones. He remarked that rejection had been neglected in the study of formal logic, and introduced a formal system to inductively derive rejections of false propositions. We call such systems refutation systems, following Scott [48] and Skura [53]. The general theory of such systems has been studied extensively by Słupecki et al. [60,61]. In this paper, we focus on refutation systems for intermediate logics.

A refutation system can be thought of as a proof system for rejection. Instead of deriving that one can correctly assert a statement through a series of truth-preserving
inferences from given axioms, as one does in a proof system of assertion, one derives
the refutability of a propositional statement through a series of non-truth preserving
inferences from given anti-axioms. Proofs in a refutation system will be called refu-
tations, and a formula will be called refutable whenever a refutation exists ending in
this formula.

Let us, by way of example, present a reformulation of the original refutation system
for the classical propositional calculus (CPC) as given by Łukasiewicz [39]. This
particular presentation, and all the following, will be in the style of Skura [54], which
goes back to Scott [48].1 Here \( x \) denotes a propositional variable, \( \phi \) and \( \psi \) both denote
propositional formulae, and \( \sigma \) denotes a substitution.

\[
\begin{align*}
\therefore \neg x & \quad \text{Ax} \\
\neg \sigma(\phi) & \quad \text{Subs} \\
\neg \phi & \quad \text{CPC} \\
\therefore \psi & \quad \text{MT}
\end{align*}
\]

This refutation system is both sound (all refutable formulae are not derivable in
CPC) and complete (all formulae that are not derivable in CPC are refutable). Skura
[56, Section 1.2] gives a thorough proof of completeness, let us simply remark that
to each classically non-derivable formula there is a substitution under which it is
equivalent to falsity, whence completeness is clear.2

Gödel [19] showed that the intuitionistic propositional calculus (IPC) enjoys the
disjunction property. With this observation in hand, Łukasiewicz [40] proposed the
following refutation system for IPC, which he conjectured to be complete.

\[
\begin{align*}
\therefore \neg x & \quad \text{Ax} \\
\neg \sigma(\phi) & \quad \text{Subs} \\
\neg \phi & \quad \text{IPC} \\
\therefore \psi & \quad \text{MT} \\
\neg \phi & \quad \text{RScott} \\
\neg \phi \vee \neg \psi & \quad \text{DP}
\end{align*}
\]

Kreisel and Putnam [36] proved that this system is not complete by constructing an
intermediate logic, a consistent axiomatic extension of IPC, now known as KP. This
logic KP has the disjunction property, and it is unequal to IPC, falsifying the conjecture.
There exist, in fact, uncountably many intermediate logics with the disjunction
property, see [5] for a survey.

Observe that the rules DP and MT are structural, in the sense that every substitution
instance of an instance of this rule is again an instance of this rule. This does not hold
for Ax and Subs, and necessarily so, as we will argue in Sect. 1.

Scott [48] gave a refutation system which is both sound and complete for IPC
by replacing DP with \( \text{RScott} \), as given in Fig. 1. The rule \( \text{RScott} \), however, has
several side-conditions which makes it inherently non-structural, in that it has instances
with invalid substitution instances. The Kleene-Aczel slash [1,34] suggests that the
refutation system obtained by replacing DP with \( \text{RKleene} \) might be complete. This
is indeed the case, as follows from de Jongh [13, Chapter IV].3 Another refutation
system was given by Dutkiewicz [14] based on the semantic tableaux of E. W. Beth.
Neither of these rules are structural.

1 See [59] for a similar presentation and further pointers to the literature.
2 See also Lemma 13 for more details.
3 We refer to Bezhanishvili [2] for a proof using the universal model, and to Iemhoff [27, Proposition 5.1]
for a proof using admissible rules. These authors do not propose such a refutation system, the rule given
here is taken from Skura [56, Section 5.4].