On constructions with 2-cardinals

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Abstract  We propose developing the theory of consequences of morasses relevant in mathematical applications in the language alternative to the usual one, replacing commonly used structures by families of sets originating with Velleman’s neat simplified morasses called 2-cardinals. The theory of related trees, gaps, colorings of pairs and forcing notions is reformulated and sketched from a unifying point of view with the focus on the applicability to constructions of mathematical structures like Boolean algebras, Banach spaces or compact spaces. The paper is dedicated to the memory of Jim Baumgartner whose seminal joint paper (Baumgartner and Shelah in Ann Pure Appl Logic 33(2):109–129, 1987) with Saharon Shelah provided a critical mass in the theory in question. A new result which we obtain as a side product is the consistency of the existence of a function \( f : [\lambda^{++}]^2 \to [\lambda^{++}]^{\leq \lambda} \) with the appropriate \( \lambda^+ \)-version of property \( \Delta \) for regular \( \lambda \geq \omega \) satisfying \( \lambda^{<\lambda} = \lambda \).

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1 Introduction

The notation used is fairly standard, for unexplained symbols and notions see [29] or [18]. If $\mu \subseteq \wp_\kappa(\lambda)$ and $X \subseteq \lambda$, then $\mu|X = \{Y \in \mu : Y \subseteq X\}$. If $X$ and $Y$ are sets of ordinals of the same order type, then $f_{YX}$ denotes the unique order preserving bijection from $X$ onto $Y$. By

$$X_1 \ast X_2$$

we mean $X_1 \cup X_2$ if $X_1, X_2$ are two sets of ordinals of the same order type and $X_1 \cap X_2 < X_1 \setminus X_2 < X_2 \setminus X_1$. Otherwise $X_1 \ast X_2$ is undefined.

Definition 1.1 ([52, 53]) Let $\kappa$ be a regular cardinal. A $(\kappa, \kappa^+)\text{-cardinal}^2$ is a family $\mu \subseteq \wp_\kappa(\kappa^+)$ which satisfies the following conditions:

1. $\mu$ is well-founded with respect to inclusion,
2. $\mu$ is locally small i.e. $|\mu|X| < \kappa$ for all $X \in \mu$,
3. $\mu$ is homogenous i.e. if $X, Y \in \mu$, $\text{rank}(X) = \text{rank}(Y)$, then $X, Y$ have the same order type and $\mu|Y = \{f_{YX}[Z] : Z \in \mu|X\}$,
4. $\mu$ is directed i.e., for every $X, Y \in \mu$ there exists $Z \in \mu$ such that $X, Y \subseteq Z$,
5. $\mu$ is locally almost directed, i.e., for every $X \in \mu$ either
   a. $\mu|X$ is directed or
   b. there are $X_1, X_2 \in \mu$ of the same rank such that
      $$X = X_1 \ast X_2 \text{ and } \mu|X = (\mu|X_1) \cup (\mu|X_2) \cup \{X_1, X_2\}$$
6. $\mu$ covers $\kappa^+$ i.e., $\cup \mu = \kappa^+$.
7. $\mu$ is neat, that is for every element $X$ of $\mu$ of nonzero rank we have
      $$X = \bigcup(\mu|X)$$

The terminology proposed here is a suggested consequence of the main point of the paper which is that the above representation of $(\kappa, 1)$-morass allows to shift the language of the theory (proofs, lemmas, theorems) into a language compatible with the part of set theory applicable in classical mathematical fields (forcing, partitions, transfinite recursion rather than the spirit of the fine structure of $L$, inner models etc.).

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1 In particular $|A|$ stands for the cardinality of $A$, $f[A]$ denotes the image of $A$ under $f$, $f \upharpoonright A$ denotes the restriction of $f$ to $A$. $A \subset B$ means the strict inclusion i.e., $A \neq B$ in that case. If $A, B$ are sets of ordinals, then ordtp($A$) denotes the order type of $A$ and we write $A < B$ if and only if $\alpha < \beta$ for all $\alpha \in A$ and $\beta \in B$. ht and rank denotes height and rank in well founded families of sets with respect to the inclusion. $\alpha <^\beta$ denotes the family of all sequences of elements from $\alpha$ of length less then $\beta$. If $\kappa$ and $\lambda$ are cardinals, then $\wp_\kappa(\lambda) = \{X \subseteq \lambda : |X| < \kappa\}$.

2 Formally, in the original terminology of [53] and [52] a $(\kappa, \kappa^+)$-cardinal is a neat simplified $(\kappa, 1)$-morass, however in many following papers e.g., [16, 21, 32] a $(\kappa, 1)$-morass is what formally Velleman called an expanded neat simplified morass. This shift towards the expanded version (already present in the above papers of Velleman) is justified by the fact that the above authors do all the calculations with the expanded versions i.e., use maps rather than sets.