An extension of Shelah’s trichotomy theorem

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Abstract Shelah (Algorithms Comb 14:420–459, 1997) develops the theory of \( \text{pcf}_I(A) \) without the assumption that \( |A| < \min(A) \), going so far as to get generators for every \( \lambda \in \text{pcf}_I(A) \) under some assumptions on \( I \). Our main theorem is that we can also generalize Shelah’s trichotomy theorem to the same setting. Using this, we present a different proof of the existence of generators for \( \text{pcf}_I(A) \) which is more in line with the modern exposition. Finally, we discuss some obstacles to further generalizing the classical theory.

Keywords pcf theory · Trichotomy · Generators

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1 Introduction

The pcf theory as presented in [6] has proven to be a powerful tool for analyzing the combinatorial structure at singular cardinals as well as their successors. Perhaps the most well-known consequence of the pcf-theoretic machinery is the following theorem due to Shelah:

\textbf{Theorem 1.1} (Shelah)

\[ \aleph_{\omega_0}^{\aleph_0} < \max \left\{ \aleph_{\omega_1}^\omega, (2^{\aleph_0})^+ \right\}. \]

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This contrasts greatly with the situation for regular cardinals, and tells us that we can get meaningful results about the power of singular cardinals in ZFC. On the other hand, we know that some of this machinery can only work for singular cardinals which are not fixed points of the $\aleph$-function. Given suitable large cardinal hypotheses, one can use Prikry-type forcings to blow up the power of some $\aleph$-fixed points to be arbitrarily large (see [3] for an overview). So if the pcf machinery can be generalized to $\aleph$-fixed points, this can only be done in a restricted manner.

In [8], Shelah does precisely this. The pcf machinery is relativized to particular ideals over some set $A$ which need not satisfy $|A| < \min(A)$. In particular, Shelah is able to obtain generators for every $\lambda \in \text{pcf}_I(A)$. The usual proof of the existence of generators requires obtaining universal cofinal sequences for each $\lambda \in \text{pcf}(A)$, and then showing that exact upper bounds for such sequences yield generators. In the classical case, one can make use of Shelah’s trichotomy theorem [6]:

**Theorem 1.2** (Chapter II, Claim 1.2 of [6]) Suppose that $\lambda$ is a regular cardinal with $\lambda > |A|^{+}$, $I$ is an ideal over $A$, and $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ is an $I$-increasing sequence of functions from $A$ to ON. Then $\vec{f}$ satisfies at least one of the following conditions:

1. **Good:** $\vec{f}$ has an exact upper bound $f \in ^{A}\text{ON}$ such that $\text{cf}(f(a)) > |A|$ for all $a \in A$.

2. **Bad:** There are sets $S(a)$ for each $a \in A$ such that $|S(a)| \leq |A|$ and an ultrafilter $D$ over $A$ disjoint from $I$ such that, for all $\xi < \lambda$, there exists some $h_\xi \in \prod_{a \in A} S(a)$ and some $\eta < \lambda$ such that $f_\xi < D h_\xi < D f_\eta$.

3. **Ugly:** There is a function $g : A \rightarrow \text{ON}$ such that, letting $t_\xi = \{ a \in A : f_\xi(a) > g(a) \}$, the sequence $\vec{t} = \langle t_\xi : \xi < \lambda \rangle$ (which is $\subseteq I$-increasing) does not stabilize modulo $I$. That is, for every $\xi < \lambda$, there is some $\xi < \eta < \lambda$ such that $t_\eta \setminus t_\xi \notin I$.

In our desired applications, the functions $\langle f_\xi : \xi < \lambda \rangle$ will belong to $\prod A$, where $A$ is a collection of regular cardinals. So if $\vec{f}$ does have an exact upper bound $f$, it would be bounded above by the function $a \mapsto a$. This means that if $f$ is Good as above, the requirement that $\text{cf}(f(a)) > |A|$ for each $a \in A$ will force that $|A| < \min(A)$. So this version of trichotomy will not work in the more general setting of [8]. While Shelah pursues a different route, it is natural to ask whether or not one can generalize the trichotomy theorem. Of course, even if we obtain this more general trichotomy theorem, we still have to show that we can find sequences that are neither bad nor ugly. Our main theorem is that one can do precisely that.

This paper is organized as follows: In Sect. 2, we extend Theorem 1.2, and show that one can still construct sequences that are neither bad nor ugly. In Sect. 3, we use this to provide a streamlined proof of the fact that generators exist for $\text{pcf}_I(A)$. Finally, we show that the no holes conclusion must fail in general, and that the standard techniques for obtaining transitive generators cannot be generalized.

**2 The trichotomy theorem**

Our goal in this section is to generalize Theorem 1.2 by replacing the assumption that $|A|^{+} < \lambda$ with assumptions about the ideal $I$ we are asking about. First, we fix some notation