N-strictness in applicative theories

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Abstract. We study the logical relationship of various forms of induction, as well as quantification operators in applicative theories. In both cases the introduced notion of N-strictness allows us to obtain the appropriate results.

1. Introduction

This paper deals with the question of the logical relationship between various forms of induction and between different quantification operators in applicative theories. These theories go back to Feferman’s systems of explicit mathematics introduced in [Fef75, Fef79]. They are based on the basic theory of operations and numbers BON introduced in [FJ93] as the classical version of Beeson’s theory EON (cf. [Bee85]) without induction. Its logic is the logic of partial terms, and the axioms comprise combinatory algebra, pairing and projection, natural numbers \( N \), definition by cases on \( N \), and primitive recursion on \( N \). For a collection of results in applicative theories and a discussion of their various applications we refer to the survey paper [JKS99].

In [FJ93] Feferman and Jäger consider two forms of induction in the context of applicative theories, set induction and formulae induction. The intermediate forms operation induction and N-induction were introduced by Jäger and Strahm in [JS96] to get an applicative theory of proof-theoretic

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strength $\varphi \omega 0$ in the presence of the quantification operator $\mu$. So it becomes a natural question how these forms of induction are logically related to each other. Especially, whether operation induction (for the formula $f \ x = 0$) is logically equivalent with N-induction (for the formula $N(f \ x)$).

It turns out that a direct translation of these two formulae is impossible. We will prove this by construction of a counter model. But asking what we would need to prove such an equivalence, the notion of N-strictness provides a solution. We call a function $f$ N-strict, if $N(f \ x)$ implies $N(x)$. The counter model shows that in BON the identity function is the only N-strict function.

Proposing a new axiom (N-Str), demanding that definition by cases is N-strict, we get translations of $f \ x = 0$ and $N(f \ x)$ into each other and therefore the equivalence of operation and N-induction. The axiom (N-Str) is satisfied in all usual models of BON, excluding only artificial ones such as the constructed counter model. Moreover, it provides several applications which make the work within applicative theories easier.

In a second part we discuss the logical relationship of three different forms of quantification operators, $\mu$, $E$, and $E^\#$. These operators allow us to eliminate quantifiers over N and test functions for a zero. $\mu$ and $E$ operate on total functions, and whereas $\mu$ returns a particular zero, if there is one, $E$ will return 0 in this case. $E^\#$ differs from $E$ by operating on partial functions also, but only if they have a zero. We investigate the logical relationship of these operators. For instance, N-strictness is needed to define $E$ from $E^\#$. Nevertheless, the totality test of $\mu$ and $E$ provides enough N-strictness to prove the equivalence of operation and N-induction.

The aim of this paper is not only to solve some particular questions about the logical relationship of various concepts, but also to give a deeper insight in the combinatorial structure of applicative theories. The plan of the paper is as follows. First we introduce the basic theory BON and give some immediate consequences, especially the definition of a term outside N, and the construction of non-strict case distinctions. Then we define the different forms of induction and state some first results. In Sect. 4 we introduce the crucial notion of N-strictness, propose the new axiom (N-Str) and use it to establish the required results. The last section is devoted to quantification operators and their logical relationship. Finally, in an appendix we give the detailed proof for the missing N-strictness in BON by constructing a suitable counter model.

2. The basic theory of operations and numbers BON

As introduced in [FJ93] the basic theory of operations and numbers BON is formulated in $\mathcal{L}_p$, the first order language of partial operations and numbers. $\mathcal{L}_p$ comprises individual variables $x, y, z, u, v, f, g, h, \ldots$ (possibly