Axiomatization of abelian-by-$G$ groups
for a finite group $G$

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Abstract. We show that, for each finite group $G$, there exists an axiomatization of the class of abelian-by-$G$ groups with a single sentence. In the proof, we use the definability of the subgroups $M^n$ in an abelian-by-finite group $M$, and the Auslander-Reiten sequences for modules over an Artin algebra.

For each group $G$ and for each property $P$ of groups, we say that a group $M$ is $P$-by-$G$ if there exists a normal subgroup $N$ of $M$ which satisfies $P$ and such that $M/N \cong G$. In a conference at the AILA-KGS Joint Meeting on Model Theory (Florence, August 1995), C. Toffalori proposed the following conjecture: if $G$ is a finite group, then the class of abelian-by-$G$ groups is elementary. He showed that this property is true, for instance, if $G$ is abelian, or if there exists no prime number $p$ such that $p^4$ divides $|G|$. The same property had been previously proved for $|G|$ square-free in [3]. In the present paper, we prove the conjecture in a stronger form:

Theorem. For each finite group $G$, there exists an axiomatization of the class of abelian-by-$G$ groups with a single sentence.

In section 1, we prove two definability properties for abelian-by-finite groups. These properties and the module-theoretic results of section 2 are used in section 3 for the proof of the theorem.

On the other hand, T. Coulbois proves in [2] that the class of group-by-$(\mathbb{Z}/2\mathbb{Z})$ groups is not elementary. Moreover, we show in section 4 that, if $G$ is the non commutative group of order 8, then the class of (nilpotent of class 2)-by-$G$ groups is not elementary.

The reader is referred to [7] concerning groups and to [6] concerning modules. Model-theoretic notions such as formula, sentence and axiomatization are also defined in [6]. For each group $M$ and for each integer $n$, we write $M^n = \langle \{ x^n \mid x \in M \} \rangle$.

1. Definable subgroups in abelian-by-finite groups

In [4], we showed that two finitely generated abelian-by-finite groups are elementarily equivalent if and only if they have the same finite images. The proof was
based on the definability of the subgroups $M^n$ for $n \geq 1$ and $M$ abelian-by-finite. A similar argument will be used here:

**Proposition 1.** Let $G$ be a finite group of order $m \geq 1$ and let $n \geq 1$ be an integer such that $G^n = 1$. Then, for each abelian-by-$G$ group $M$, any element of $M^n$ can be written as $x_1^n \cdots x_{2m-1}^n$ with $x_1, \ldots, x_{2m-1} \in M$.

**Remark.** In [5], F. Point proves that, for any integers $m, n \geq 1$, there exists an integer $k(m, n)$ such that, for each group $G$ of order $m$ and for each abelian-by-$G$ group $M$, each element of $M^n$ is a product of $k(m, n)$ $n$-th powers.

**Proof.** We consider a normal abelian subgroup $A$ of $M$ such that $M/A \cong G$, and some elements $g_1, \ldots, g_m \in M$, with $g_1 = 1$, such that $M = g_1 A \cup \cdots \cup g_m A$. We have $M^n \subset A$, and therefore $M^n$ is abelian. Any element of $M^n$ can be written as $x = (g_{r_1} a_1^n \cdots g_{r_k} a_k^n)$ with $r_1, \ldots, r_k \in \{1, \ldots, m\}$ and $a_1, \ldots, a_k \in A$. We also have $x = \prod_{1 \leq i \leq m} (\prod_{1 \leq j \leq s(i)} (g_i a_i^j)^s)$ with $a_i j \in A$ for $1 \leq i \leq m$ and $1 \leq j \leq s(i)$, since the elements $(g_i a_i^j)$ and $(g_i b_i^j)$ for $a, b \in A$ and $i, j \in \{1, \ldots, m\}$ commute.

For each $i \in \{1, \ldots, m\}$ and each $a \in A$, we have $(g_i a_i)^n = g_i^n a_i^n (a_i)^n$ with $x^n = (g_i a_i)^n = a_i^n (a_i x)^n a_i^{-n}$ for $a \in A$ and $x \in M$. Using the fact that the elements $a^n$ for $a \in A$ and $y \in M$ belong to the abelian subgroup $A$, and the identity $(uv)^n = u^n v^n$, we obtain $a_x(ab) = a_x(a)a_x(b)$ for $a, b \in A$ and $x \in M$.

Now, for $1 \leq i \leq m$, we have

$$\prod_{1 \leq i \leq s(i)} (g_i a_i^j)^n = \prod_{1 \leq j \leq s(i)} (g_i^n a_i^n (a_i)^n)^n = g_i^{ns(i)} a_i^{ns(i)} = g_i^{-n} (g_i a_i^{1 \cdots s(i)})^n$$

Consequently, $\prod_{1 \leq i \leq s(i)} (g_i a_i^j)^n$ is a $n$-th power if $i = 1$, and a product of two $n$-th powers otherwise.

**Proposition 2.** For each integer $k \geq 2$ and for each finite group $F$ of order $k$, there exist some formulas $\varphi_F(a, u_1, \ldots, u_{k-1})$ and $\psi_F(a_1, \ldots, a_{k-1})$ such that:

1) For each group $M$ and for each maximal abelian normal subgroup $A$ of $M$ such that $M/A \cong F$, there exist some elements $a_1, \ldots, a_{k-1} \in A$ such that $M$ satisfies $\psi_F(a_1, \ldots, a_{k-1})$ and $A = \{x \in M \mid \varphi_F(x, a_1, \ldots, a_{k-1})\}$.

2) For each group $M$ and for any elements $a_1, \ldots, a_{k-1} \in M$, if $M$ satisfies $\psi_F(a_1, \ldots, a_{k-1})$, then $A = \{x \in M \mid \varphi_F(x, a_1, \ldots, a_{k-1})\}$ is a maximal abelian normal subgroup of $M$ which contains $a_1, \ldots, a_{k-1}$, and we have $M/A \cong F$.

**Proof.** The first part of Proposition 2 essentially comes from C. Toffalori’s conference. For each integer $k \geq 2$, we denote by $\varphi_F(u, u_1, \ldots, u_{k-1})$ the formula

$$(\forall v) (\forall_{1 \leq i < k-1} u^v u_i = u_i u^v) \land (\forall v \forall w) (u^v u^w = u^w u^v).$$