Some remarks on a question of D. H. Fremlin regarding $\epsilon$-density

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Abstract. We show the relative consistency of $\aleph_1$ satisfying a combinatorial property considered by David Fremlin (in the question DU from his list) in certain choiceless inner models. This is demonstrated by first proving the property is true for Ramsey cardinals. In contrast, we show that in ZFC, no cardinal of uncountable cofinality can satisfy a similar, stronger property. The questions considered by D. H. Fremlin are if families of finite subsets of $\omega_1$ satisfying a certain density condition necessarily contain all finite subsets of an infinite subset of $\omega_1$, and specifically if this and a stronger property hold under $\text{MA} + \neg\text{CH}$. Towards this we show that if $\text{MA} + \neg\text{CH}$ holds, then for every family $\mathcal{A}$ of $\aleph_1$ many infinite subsets of $\omega_1$, one can find a family $\mathcal{S}$ of finite subsets of $\omega_1$ which is dense in Fremlin’s sense, and does not contain all finite subsets of any set in $\mathcal{A}$.

We then pose some open problems related to the question.

1. Introduction

Definition 11. Let $0 < \epsilon \leq 1$ be fixed but arbitrary, and let $\kappa$ be an infinite cardinal. Call a family $\mathcal{S} \subseteq [\kappa]^{<\aleph_0}$ $\epsilon$-dense open iff $\mathcal{S}$ is downward closed, i.e., $p \in \mathcal{S}$ and $q \subseteq p$ implies $q \in \mathcal{S}$, and $\mathcal{S}$ in addition satisfies the property

$$\forall p \in [\kappa]^{<\aleph_0} \exists q \subseteq p \exists q \in \mathcal{S} \land |q| \geq \epsilon \cdot |p|.$$ 

D. H. Fremlin in question DU from [5] asks if $\kappa = \aleph_1$ and we have $\epsilon$ and $\mathcal{S}$ as in the above, then does there necessarily exist $A \in [\aleph_1]^{\aleph_0}$ so that $[A]^{<\aleph_0} \subseteq \mathcal{S}$? In fact this particular question is credited to S. Argyros, while the original question of D. H. Fremlin is if under $\text{MA} + \neg\text{CH}$ one can find an uncountable such $A$. For the motivation behind the question, we refer the reader to Fremlin’s note [6].

The reason one would ask the question in the above form is that an apparently folklore argument (communicated to us by Ilijas Farah) shows that $\text{CH}$ implies that there is $\mathcal{S}$ on $\aleph_1$ which is $1/2$-dense open, but contains no $[A]^{<\aleph_0}$ for an uncountable $A$, while an easy modification of an argument by P. Erdős and R. Rado in [3] answers negatively the analogue of Fremlin’s question with $\aleph_0$ in place of $\aleph_1$ (see the appendix for these facts).

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Ilijas Farah brought to our attention the fact that $\epsilon$-density is a variation on Kelley’s intersection number [9] of a family of sets, but that if in question DU, instead of requiring that $\mathcal{S}$ be $\epsilon$-dense open, we ask that $\mathcal{S}$ be a family of finite sets obtained from some family with the intersection number positive, then the problem has a positive solution — there is an infinite set with all finite subsets in $\mathcal{S}$, and under $MA(\aleph_1)$ there is an uncountable such set. Note [6] by D. H. Fremlin deals with certain minimal $\epsilon$-dense families and furthers the connection with intersection numbers.

In Section 2, we will define the notions of a cardinal $\kappa$ being Fremlin and being strongly Fremlin. We show that modulo the existence of a Ramsey cardinal, it is consistent that $\aleph_1$ is strongly Fremlin in a model of $ZF$ which satisfies $DC$. We then show in Section 3 that if the original condition in question DU is fortified, no cardinal of uncountable cofinality can be Fremlin in this stronger sense, in a model of $ZFC$. In Section 4 we show that under $MA (+\neg CH)$, for every family $A$ of $\aleph_1$ elements of $[\omega_1 \cdot \aleph_0$, there is a $1/2$-dense open family $S$ of finite subsets of $\omega_1$ such that for no $A \in A$ do we have that $[A]^{<\aleph_0} \subseteq \mathcal{S}$. The paper ends by giving further open problems.

2. Choiceless universes

We begin by noting that if $\kappa$ is a Ramsey cardinal, then the above property holds for $\kappa$. Specifically, we show the following.

**Fact 21.** Suppose $\kappa$ is a Ramsey cardinal and $\mathcal{S}$ is $\epsilon$-dense open for an $\epsilon \in (0, 1]$. There is then $A \in [\kappa]^\kappa$ such that $[A]^{<\aleph_0} \subseteq \mathcal{S}$.

**Proof.** Define $f : [\kappa]^{<\aleph_0} \to 2$ by $f(p) = 1$ iff $p \in \mathcal{S}$. Since $\kappa$ is a Ramsey cardinal, let $A \subseteq \kappa$, $A \in [\kappa]^\kappa$ be such that $A$ is homogeneous for $f$. We show that $A$ is our desired set by considering the following two cases.

1. There are unboundedly many $n < \omega$ so that $f^n[A]^n = \{1\}$, i.e., so that $[A]^n \subseteq \mathcal{S}$. Since $\mathcal{S}$ is downward closed, we automatically have that $[A]^{<\aleph_0} \subseteq \mathcal{S}$.

2. Case 1 doesn’t hold, i.e., there is $n_0 < \omega$ so that for all $n \geq n_0$, we have $f^n[A]^n = \{0\}$. This means that if $n \geq n_0$ and $p \in [A]^n$, then $p \notin \mathcal{S}$. In this situation, let $m > n_0$ be large enough so that $m \cdot \epsilon > n_0$, and let $p \in [A]^m$.

By the properties of $\mathcal{S}$, there must be $q \subseteq p$, $q \in \mathcal{S}$ so that $|q| \geq \epsilon \cdot |p|$, i.e., $|q| \geq m \cdot \epsilon > n_0$. As $f(q) = 1$, this contradicts that $q \in [A]^\ell$ for some $\ell > n_0$, i.e., $f(q) = 0$.

The contradiction just obtained means that Case 2 above can’t hold, i.e., that the homogeneous set $A$ for $f$ has the desired properties. This completes the proof of Fact 21.

More generally,

**Definition 22.** An infinite cardinal $\kappa$ is called Fremlin if for every $\epsilon$-dense open family $\mathcal{S}$ of finite subsets of $\kappa$, there is an infinite $A \subseteq \kappa$ all of whose finite subsets are contained in $\mathcal{S}$. If we can always guarantee that such an $A$ can be found with $|A| = \kappa$, we say that $\kappa$ is strongly Fremlin.