Max and min limiters

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Abstract. If $A \subseteq \omega$, $n \geq 2$, and the function $\max(\{x_1, \ldots, x_n\} \cap A)$ is partial recursive, it is easily seen that $A$ is recursive. In this paper, we weaken this hypothesis in various ways (and similarly for “min” in place of “max”) and investigate what effect this has on the complexity of $A$. We discover a sharp contrast between retraceable and co-retraceable sets, and we characterize sets which are the union of a recursive set and a co-r.e., retraceable set. Most of our proofs are noneffective. Several open questions are raised.

1. Introduction

It is easy to see that, for $A \subseteq \omega$ and $n \geq 2$, if the (partial) function $\max(\{x_1, \ldots, x_n\} \cap A)$ is partial recursive, then $A$ is recursive. Suppose, on the other hand, that one can effectively eliminate a possibility for $\max(\{x_1, \ldots, x_n\} \cap A)$, specifically, that one can, for any $n$ distinct natural numbers $x_1, \ldots, x_n$, calculate an $i$, $1 \leq i \leq n$, such that $x_i \neq \max(\{x_1, \ldots, x_n\} \cap A)$. What does this tell us about $A$? In particular, must $A$ be recursive? Suppose that one then weakens the assumption to the existence of a recursive function $g$ such that, for all $x_1, \ldots, x_n$, $W_{g(x_1, \ldots, x_n)}$ is a proper subset of $\{x_1, \ldots, x_n\}$ containing $\max(\{x_1, \ldots, x_n\} \cap A)$. Do we now have more possibilities for the complexity of $A$? In this paper we consider questions of this sort, including analogous questions regarding “min” in place of “max.” In many cases we find that the methods we use are at least as interesting as the statements of the theorems.

2. Background material

Definition 2.1. Let $n, k \geq 1$, and let $f: \omega^k \rightarrow \omega$.

1. $f \in \text{EN}(n)$ if there exist $n$ partial recursive functions $\varphi_1, \ldots, \varphi_n$ such that, for all $x_1, \ldots, x_k$, there exists an $i$, $1 \leq i \leq n$, such that $\varphi_i(x_1, \ldots, x_k)$ converges...
to \( f(x_1, \ldots, x_k) \). If \( f \in \text{EN}(n) \), we say that \( f \) is \( n \)-enumerable. Note that \( f \) is recursive iff \( f \) is 1-enumerable.

2. \( f \in \text{SEN}(n) \) if there exist \( n \) total recursive functions \( g_1, \ldots, g_n \) such that, for all \( x_1, \ldots, x_k \), there exists an \( i \), \( 1 \leq i \leq n \), such that \( g_i(x_1, \ldots, x_k) = f(x_1, \ldots, x_k) \). If \( f \in \text{SEN}(n) \), we say that \( f \) is strongly \( n \)-enumerable.

**Definition 2.2.** Let \( A \subseteq \omega \), and \( n \geq 1 \).

1. \( C_n^A : \omega^n \to \{0, 1\}^n \) is the function defined by
   \[
   C_n^A(x_1, \ldots, x_n) = (\chi_A(x_1), \ldots, \chi_A(x_n)),
   \]
   where \( \chi_A \) is the characteristic function of \( A \).

2. \( \#_n^A : \omega^n \to \{0, \ldots, n\} \) is the function defined by
   \[
   \#_n^A(x_1, \ldots, x_n) = |\{i : 1 \leq i \leq n \land x_i \in A\}|.
   \]

In his PhD thesis, Stanford University, 1987, Richard Beigel proved the following interesting theorem.

**Theorem 2.3 (Nonspeedup Theorem [1,2]).** If \( (\exists n \geq 1)[C_n^A \in \text{EN}(n)] \), then \( A \) is recursive.

Beigel conjectured that if \( \#_n^A \in \text{EN}(n) \) for some \( n \geq 1 \), then \( A \) is recursive. In 1987, Owings proved the following weak form of Beigel’s conjecture.

**Theorem 2.4 (Weak Cardinality Theorem [14]).** If \( (\exists n \geq 1)[\#_n^A \in \text{SEN}(n)] \), then \( A \) is recursive.

Using entirely different methods, Martin Kummer proved Beigel’s conjecture in 1990.

**Theorem 2.5 (Cardinality Theorem [9]).** If \( (\exists n \geq 1)[\#_n^A \in \text{EN}(n)] \), then \( A \) is recursive.

Kummer’s proof of the Cardinality Theorem rested on the next two lemmas.

**Definition 2.6.** For \( n \in \omega \), \( B_n \) is the full binary tree of height \( n \).

**Lemma 2.7 ([9]).** Let \( T \) be a subtree of \( 2^{\omega} \), the full binary tree with \( \omega \) levels, and let \( n \geq 1 \). If \( B_{n-2} \) can be embedded in \( T \), then there exist natural numbers \( x_1, \ldots, x_n \), nodes \( \tau_1, \ldots, \tau_{n+1} \) of \( T \), and \( b \in \{0, 1\} \) such that, for \( i, j \) with \( 1 \leq i \leq n \) and \( 1 \leq j \leq n + 1 \),

\[
\tau_j(x_i) = \begin{cases} 
1 - b, & \text{if } 1 \leq i < j; \\
b, & \text{if } j \leq i \leq n.
\end{cases}
\]

**Lemma 2.8 (R.E. Tree Lemma [9]).** Let \( T \) be an r.e. subtree of \( 2^{\omega} \). If, for some \( m \geq 1 \), \( B_m \) cannot be embedded in \( T \), then every infinite branch of \( T \) is recursive.

In 1972, Jockusch and Soare proved the following important theorem.