Viscous forces on nematic defects

Abstract The effect of backflow in defect dynamics is assessed by computing the viscous force on point and line defects that move due to reorientation. It is found that defects with a positive winding number are accelerated while defects with a negative winding number are slowed down by the backflow. The results are in agreement with experimental and numerical results for defect annihilation.

Keywords Liquid crystals · Defect dynamics · Backflow

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1 Introduction

Defects arise in many different areas of physics, such as condensed matter systems, particle physics, and cosmology [1]. They are often found in connection with a symmetry breaking phase transition, and so it is no surprise that the numerous liquid crystalline phases show characteristic types of defects. Indeed, liquid crystals offer a good opportunity to explore various aspects of defect dynamics in the laboratory. Possibly the simplest system is a nematic liquid crystal, a fluid consisting of effectively uniaxial molecules that tend to align their long axes in a common direction. It is usually described by a unit vector field $\mathbf{n}$, the nematic director, that indicates the local average orientation.

An unexpected observation in the annihilation of defect pairs in nematic liquid crystals has in recent years sparked a lot of interest in this particular process. Cladis and Brand [2] have reported very different velocities for $a + 1$ and $a - 1$ defects while they approach each other, so that their annihilation point is not in the middle between their starting positions: the $+1$ defect moves several times faster than the $-1$ defect. A similar effect has been seen in smectic films, and the influence of backflow has been suggested as a possible explanation [3]. The importance of backflow has subsequently been confirmed by numerical investigations [4–6] based on models with a variable order parameter. However, an explanation of the effect in terms of the basic director model is still missing.

The dynamical theory for nematic liquid crystals goes back to the works of Ericksen [7] and Leslie [8], and it has been successfully applied to many practical situations. However, the dynamics of defects in nematics still poses major problems. The intricate interplay between flow and orientation allows for analytical solutions only for the simplest problems, and numerical methods also fail in the presence of defects [9]. The main problem lies in the validity of numerical schemes in the vicinity of defects where the director field and its derivatives diverge. It is possible to avoid these problems by using theories of the Landau-Ginzburg-deGennes
type, but this approach leads to the loss of scaling invariance. This is a severe drawback, because the small size of the defect core determines the length scale that can be investigated. Indeed, the numerical results on defect annihilation reported in [4–6] are valid only for defect distances in the sub-micron range and hence do not apply directly to experimentally accessible scenarios.

In this paper, we show that the observed asymmetry can be explained within classical Ericksen-Leslie theory. Defect dynamics is governed by two types of forces: elastic forces that stem from the free energy stored in the director field, and viscous forces related to the energy dissipated in the process. The elastic forces were closely investigated in [10], and it was found that the elastic stress determines the forces that drive the defects even when flow is completely neglected. However, elastic forces are not sufficient to explain the asymmetry in defect annihilation [10, 11], where backflow plays a crucial role [4–6].

To assess the effect of viscous forces in defect dynamics, we consider a single defect that moves due to reorientation in an environment that is initially at rest. The director reorientation creates a viscous stress that is responsible for the onset of backflow. Though this model does not lead to a solution of the time dependent Ericksen-Leslie equations, it yields the viscous force created by a moving defect.

In Sect. 2, the relevant aspects of Ericksen-Leslie theory are recalled and the modelling assumptions and their consequences are outlined. In Sect. 3, the viscous forces that arise in the motion of line defects of arbitrary winding number are computed. In Sect. 4, we look at the two simplest point defects, a radial and a hyperbolic hedgehog. For both line and point defects it is found that defects with negative winding number are slowed down by the viscous forces, while defects with positive winding number are accelerated. The main conclusions of this work are drawn in Sect. 5.

2 Forces on defects

Defects in nematic liquid crystals are locations where the nematic director \( \mathbf{n} \) is undefined and the elastic energy density \( \sigma(\mathbf{n}, \nabla \mathbf{n}) \) diverges. We use the simplest form of the elastic energy density, the one-constant approximation:

\[
\sigma = \frac{K}{2} |\nabla \mathbf{n}|^2, \tag{1}
\]

with this energy the equilibrium director field of both point and line defects take fairly simple forms.

To assess the role of backflow in the motion of defects, one has to consider the material flow velocity \( \mathbf{v} \). It is determined by the momentum balance, which in the absence of body forces takes the form

\[
\frac{d}{dt} \int_C \rho \mathbf{v} d\mathbf{v} = \int_{\partial C} \mathbf{T} \cdot \mathbf{n} d\mathbf{a}. \tag{2}
\]

Here \( C \) is a volume with boundary \( \partial C \) and outer unit normal \( \mathbf{n} \), \( \rho \) is the mass density, and \( \mathbf{T} \) is the stress tensor. Away from the defect, the divergence theorem can be applied to find the momentum balance in point form,

\[
\rho \mathbf{\dot{v}} = \text{div} \mathbf{T}, \tag{3}
\]

where the dot denotes the material time derivative. For a smooth vector field \( \mathbf{a} \) this is given by

\[
\dot{\mathbf{a}} = \frac{\partial}{\partial t} \mathbf{a} + (\nabla \mathbf{a}) \mathbf{v}. \tag{4}
\]

The right-hand side of Eq. (3) is the force density at any regular point in the liquid crystal. The force on the defect itself is defined by the right-hand side of (2) when \( C \) is a suitably small neighbourhood of the defect which is to be interpreted as the defect core. This core marks the limit of validity of the director continuum theory.

In a liquid crystal, the stress tensor is the sum

\[
\mathbf{T} = \mathbf{T}^{(v)} + \mathbf{T}^{(e)} - p \mathbf{I} \tag{5}
\]

of a viscous part \( \mathbf{T}^{(v)} \), an elastic part \( \mathbf{T}^{(e)} \), and a contribution \(-p \mathbf{I}\) from the hydrostatic pressure \( p \) that arises from the incompressibility condition

\[
\text{div} \mathbf{v} = 0. \tag{6}
\]