A generalized continuum theory with internal corner and surface contact interactions

To David Steigmann in recognition of the depth and breadth of his research contributions and his unwavering dedication to the advancement of fundamental continuum mechanics.

Abstract We consider a classical derivation of a continuum theory, based on the fundamental balance laws of mass and momenta, for a body with internal corner and surface contact interactions. The balances of mass and linear and angular momentum are applied to the arbitrary parts of a continuum which supports non-classical internal corner and surface contact interactions. The form of the specific corner contact interaction force measured per unit length of the corner is derived. A generalized form of Cauchy’s stress theorem is obtained, which shows that the surface traction on an oriented surface depends in a specific way on both the oriented unit normal as well as the curvature of the surface. An explicit form of the surface-couple traction which acts on every oriented surface is obtained. Two fields in the continuum, which are denoted as stress and hyperstress fields, are shown to exist, and their role in representing the surface traction and the surface-couple traction is identified. Finally, the field equations for this theory are determined, and a fundamental power theorem is derived. In the absence of internal corner and surface-couple traction interactions, the equations of classical continuum mechanics are recovered. There is no appeal to any ‘principle of virtual power’ in this work.

Keywords Generalized Cauchy stress theorem · Non-simple materials · Continuum mechanics · Generalized continuum

Mathematics Subject Classification 74A05 · 74A10 · 74A30 · 74A35

Contents

1 Setting and introduction ................................................. 276
  1.1 Setting, viewpoint and motivation .................................. 276
  1.2 Introduction ..................................................... 277
2 Fundamental fields .................................................... 278
  2.1 Mass ............................................................... 278
  2.2 Forces and couples ................................................ 278
3 Balance laws and consequences ........................................ 281
  3.1 Oriented smooth surface .......................................... 281
  3.2 Oriented cornered surface ...................................... 286
  3.3 Arbitrary material part .......................................... 287
4 The power theorem .................................................... 290
5 Synopsis .............................................................. 291

Communicated by Victor Eremeyev, Peter Schiavone and Francesco dell’Isola.

R. Fosdick (✉)
Department of Aerospace Engineering and Mechanics, University of Minnesota, Minneapolis, MN 55455, USA
E-mail: fosdick@aem.umn.edu
1 Setting and introduction

1.1 Setting, viewpoint and motivation

In recent years, there has been a renewed interest in the development of continuum mechanics theories for materials which exhibit multiple length scales by the use of the so-called principle of virtual power as applied to arbitrary parts of the body. From a fundamental point of view, I have generally found the developments somewhat arbitrary and principally unmotivated: In some cases, versions of the ‘principle’ have been stated for arbitrary parts of a body, but applied only to parts with smooth boundaries. However, when arbitrary is taken seriously, large deficiencies appear as a result of underrepresenting the importance of the existence of non-classical corner contact interactions in addition to the classical surface contact interactions (see [2,3] for a related discussion). Moreover, many statements of the ‘principle’ include the a priori introduction of the four primitive physical fields of surface traction, stress, surface hypertraction and hyperstress throughout the body (see, for example [5,6,8,10]). For example, in the context of classical continuum mechanics, the existence and concept of a surface traction vector as a force per unit area on any oriented surface through a point of the distorted state of a body and the independent existence and concept of a stress tensor at the same point are both presumed. However, by name, there is implication that they both represent a force per unit area, and one is left pondering about the difference. It would seem that one of these two fields should be a primitive of the theory, and the other should be derived from it. The same issue arises in relation to the hypertraction and hyperstress fields.

Aside from the ‘principle of virtual power,’ fundamental internal edge interactions and their relation to surface tension have been considered in [9] with a view toward generalizing Cauchy’s stress theorem so as to include surface curvature as well as surface unit normal in the representation of surface traction on oriented surfaces within a body. This idea is further improved upon in [1], but I can find no other works where the notion of hyperstress is derived from hypertraction.

My interpretation of those primitive elements that are introduced in the ‘principle of virtual power’ for arbitrary parts of a body is that the so-called stress and hyperstress fields might be better reinterpreted as specific energy and specific hyperenergy tensor fields measured per unit volume and per unit volume/length, respectively—fundamental tensor fields that simply are assumed to exist and introduced in order to express the ‘principle’ as a hypothesis of balance between the internal virtual power and the external virtual power for any part. The existence of the fields that characterize stress and hyperstress then should come from derivation and appropriate definition related to surface traction and surface hypertraction.

To emphasize, consider, for example, the ‘principle’ as applied to classical continuum mechanics. It suffices for the point I wish to make to drop the body force field and to assume that the body is static. In this case, the ‘principle’ states

$$\int_{P} \mathbf{T} \cdot \nabla \mathbf{u} = \int_{\partial P} \mathbf{t} \cdot \mathbf{n} \quad \forall \mathbf{P} \subseteq B \quad \text{and} \quad \forall \mathbf{u}. \quad (1.1)$$

Here, the field $\mathbf{T}$ is hypothesized to exist as a suitable specific energy volumetric field so that (1.1) may hold; the surficial traction field $\mathbf{t} = \mathbf{t}(x, n)$ is introduced as a primitive field along with its traditional physical interpretation for all $x \in B$ and all $n \in \text{Unit}$. In (1.1), it is understood that $\mathbf{t} = \mathbf{t}(x, \mathbf{n}(x))$, where $\mathbf{n}(x)$ is the outer unit normal to $\partial P$.

Now, an application of the divergence theorem and a standard calculus of variations argument, which only consider smooth $P$ in (1.1), readily yield the conclusion

$$\mathbf{t} = \mathbf{t}(x, n) = \tilde{T}(x)n, \quad \text{div} \tilde{T}(x) = 0 \quad \forall x \in B \quad \text{and} \quad \forall n \in \text{Unit}. \quad (1.2)$$

Thus, as is claimed in the literature, the ‘principle’ proves the fundamental Cauchy theorem and identifies the field $\mathbf{T}$ as the Cauchy stress tensor, which satisfies the local differential equation of force balance.

A more fundamental classical argument, which recognizes the condition (1.1) as a statement of balance between the volume integral of a density function and the corresponding boundary surface integral of a density function for arbitrary volumes, readily yields, by localization, the existence of a Cauchy stress tensor field $\mathbf{T} = \mathbf{T}(x)$ for all $x \in B$ such that

$$\mathbf{t} = \mathbf{t}(x, n) = \mathbf{T}(x)n \quad \forall x \in B \quad \text{and} \quad \forall n \in \text{Unit}. \quad (1.3)$$

Substitution of (1.3) into (1.1) and use of the divergence theorem to replace the surface integral by a volume integral, together with the subsequent use of the arbitrariness of the variation $\mathbf{u}$, finally yield the following identification of $\mathbf{T}$ as well as the local differential equation of force balance: