Rigorous derivation of the effective model describing a non-isothermal fluid flow in a vertical pipe filled with porous medium

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Abstract This paper reports an analytical investigation of non-isothermal fluid flow in a thin (or long) vertical pipe filled with porous medium via asymptotic analysis. We assume that the fluid inside the pipe is cooled (or heated) by the surrounding medium and that the flow is governed by the prescribed pressure drop between pipe’s ends. Starting from the dimensionless Darcy–Brinkman–Boussinesq system, we formally derive a macroscopic model describing the effective flow at small Brinkman–Darcy number. The asymptotic approximation is given by the explicit formulae for the velocity, pressure and temperature clearly acknowledging the effects of the cooling (heating) and porous structure. The theoretical error analysis is carried out to indicate the order of accuracy and to provide a rigorous justification of the effective model.

Keywords Darcy–Brinkman–Boussinesq model · Thin pipe · Newton cooling condition · Asymptotic approximation · Error estimates

1 Introduction

The study of fluid flow in porous media has primarily been initiated by the situations naturally arising in geophysical systems. Since the recognition of engineering importance of the porous media flow, the investigation of heat transfer phenomena through fluid-saturated porous media has become a “hot” area of research, see, for example, [33] and the references therein. This area of research has been motivated by a broad range of engineering applications, such as catalytic reactors, geothermal systems, heat pipes, packed beds, fluidized beds. In the present paper, we address an important part of the above-mentioned applications, namely the fluid flow in a cooled (or heated) pipe filled with sparsely packed porous medium.

Different empiric laws are employed to describe the filtration of a fluid through porous medium. The majority of the existing studies related to the heat and fluid flow through porous media make use of the Darcy law [10] stating that the filtration velocity is proportional to the driving pressure gradient. However, being a first-order partial differential equation for the velocity, Darcy law cannot handle the no-slip boundary condition imposed on an impermeable wall and that represents its major drawback. The Brinkman-extended [6] Darcy flow model removes this deficiency by adding a new second-order viscous term to a Darcy law.

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The justification of its usage in describing the porous media flows has been provided in many papers, see, for example, [1, 21, 24, 35]. In particular, the Darcy–Brinkman model is amenable for numerical simulations since it allows to resolve problems with physically relevant boundary conditions on impermeable boundary.

The thermodynamic closure of the Darcy–Brinkman model is established by adding the heat conduction equation which is coupled to the momentum equation through additional gravitational term on the right-hand side. For thermal convection to occur, the density of the fluid appearing in gravitational term must be a function of temperature. Such complex system of partial differential equations needs to be simplified for the sake of analytical investigation, and the most useful tool is to employ the Boussinesq approximation [5]. It is based on the assumption that the variations of the fluid density can be ignored everywhere except in the vital buoyancy term involving the thermal expansion coefficient. As a consequence, the equation of continuity reduces to a divergence-free condition, just as for an incompressible fluid. Furthermore, all physical properties of the fluid are set to be constant and the gravitational force is assumed to depend linearly on the fluid temperature. Thus, in the sequel we analyze the Darcy–Brinkman–Boussinesq system written in the non-dimensional form as:

\[
\begin{align*}
\mathbf{u} - \tilde{D} a \Delta \mathbf{u} + \nabla p &= Ra D T e_3, \\
\text{div} \mathbf{u} &= 0, \\
- \Delta T + \mathbf{u} \cdot \nabla T &= 0.
\end{align*}
\]  

Here \( \mathbf{u} \) and \( p \) denote (non-dimensional) seepage velocity and kinematic pressure, respectively, while \( T \) is the non-dimensional temperature of the fluid. The characteristic parameters of the system are the Brinkman–Darcy number \( \tilde{D} a \) and Rayleigh–Darcy number \( Ra D \) (see Sect. 2). \( e_3 \) denotes the unit normal vector directed upward. For a derivation and detailed physical background of the above system we refer to [33]. We also refer the reader to [36] where interesting generalizations of the Darcy–Brinkman model have been obtained using general thermodynamic framework.

In this work, we study the flow governed by (1–3) in a thin (or long) vertical pipe with (constant) circular cross section:

\[
\Omega^\epsilon = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x' = (x_1, x_2) \in \epsilon B, 0 < x_3 < 1 \}.
\]  

Here \( B = B(0, 1) \subset \mathbb{R}^2 \) is the unit circle, while \( \epsilon \ll 1 \) is the small (non-dimensional) parameter denoting the ratio between pipe’s thickness and its length. Motivated by the above-mentioned applications, we assume that the fluid inside the pipe is cooled (or heated) by the surrounding medium and that the flow is governed by a prescribed pressure drop. In view of that, the system (1–3) is endowed with the mixed Dirichlet–Robin boundary conditions for the temperature and the pressure boundary condition prescribed at pipe’s ends (see Sect. 2). It is obvious that one cannot hope to derive the exact solution of such nonlinear problem so we aim to derive an asymptotic approximation of the solution. Observing that the magnitude of some characteristic parameters can have a serious impact on the effective behavior of the flow, we compare those parameters with small parameter \( \epsilon \) and construct, in the critical case, the two-scale asymptotic expansion of the solution in terms of \( \epsilon \). The main difficulty arises from the fact that the governing system is coupled so we have to simultaneously solve boundary value problems for the velocity and temperature. Nevertheless, we manage to explicitly compute the approximation for the velocity, pressure and temperature distribution (see Sect. 3). The effects of the cooling (heating) process at the pipe’s lateral boundary and porous structure inside the pipe are clearly visible in the formally derived asymptotic solution. Last but not least, we rigorously justify our effective model by evaluating the difference between the exact solution (which cannot be found) and the asymptotic one in the appropriate functional norm (see Sect. 4). By doing that, we provide the precise order of accuracy of the constructed approximation and that represents our main contribution.

We conclude the introduction by adding more bibliographic remarks on the subject. The isothermal porous medium flow has been successfully analyzed in numerous papers, both analytically (see, for example, [14, 29, 31, 32, 34, 39]) and numerically (see, for example, [16, 37, 38]). If temperature variations of the fluid are not to be neglected, one cannot find so many results throughout the literature. Interesting mathematical analyses on non-isothermal porous medium flows can be found in [3, 20], and they merit careful reading. In case of fully developed (coupled) flow in a pipe, the investigations are carried out only numerically, we refer the reader to [8, 19]. Analytical results have been reported only for decoupled problems, mostly in 2D channels, see [7, 17, 18]. This means that the velocity distribution is, in fact, known in the heat conduction equation (3) making the problem less challenging from the analytical point of view. Furthermore, no heat exchange on the domain’s boundary has been considered in those papers. Thus, the goal of the present paper is to take