Stability of elliptical vortices from “Imperfect–Velocity–Impulse” diagrams

Abstract In 1875, Lord Kelvin proposed an energy-based argument for determining the stability of vortical flows. While the ideas underlying Kelvin’s argument are well established, their practical use has been the subject of extensive debate. In a forthcoming paper, the authors present a methodology, based on the construction of “Imperfect–Velocity–Impulse” (IVI) diagrams, which represents a rigorous and practical implementation of Kelvin’s argument for determining the stability of inviscid flows. In this work, we describe in detail the use of the theory by considering an example involving a well-studied classical flow, namely the family of elliptical vortices discovered by Kirchhoff. By constructing the IVI diagram for this family of vortices, we detect the first three bifurcations (which are found to be associated with perturbations of azimuthal wavenumber $m = 3, 4$ and $5$). Examination of the IVI diagram indicates that each of these bifurcations contributes an additional unstable mode to the original family; the stability properties of the bifurcated branches are also determined. By using a novel numerical approach, we proceed to explore each of the bifurcated branches in its entirety. While the locations of the changes of stability obtained from the IVI diagram approach turn out to match precisely classical results from linear analysis, the stability properties of the bifurcated branches are presented here for the first time. In addition, it appears that the $m = 3, 5$ branches had not been computed in their entirety before. In summary, the work presented here outlines a new approach representing a rigorous implementation of Kelvin’s argument. With reference to the Kirchhoff elliptical vortices, this method is shown to be effective and reliable.

Keywords Vortex dynamics · Stability · Kirchhoff elliptical vortices · Kelvin’s argument

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1 Introduction

More than a century ago, Lord Kelvin proposed that a steady vortical flow realizes an extremum of the kinetic energy, for a given impulse [22]. It appears that Kelvin regarded this proposition as being self-evident, as he provided no proof for it; the first analytical confirmation is instead due to Benjamin in 1976 [1]. The argument can be illustrated as follows, with reference to a vortex configuration rotating at a rate $\Omega$.
vorticity-preserving perturbations, a steady two-dimensional flow is associated with a stationary point of the functional $H$:

$$H = E - \Omega J,$$

(1)

where $E$ and $J$ are the excess kinetic energy and angular impulse, respectively, given by:

$$E = -\frac{1}{4\pi} \iiint \omega(x) \omega(x') \log |x - x'| \, dx \, dy \, dx' \, dy'$$

(2)

$$J = -\frac{1}{2} \iint \omega(x) |x|^2 \, dx \, dy.$$ 

(3)

In the above, $\omega$ is the vorticity, while the integrals are taken over all space. Since $E$ and $J$ are conserved in an inviscid fluid, while $\Omega$ is treated as a fixed parameter, $H$ is also a conserved quantity. If the stationary point is a maximum or a minimum in the solution space (implying that the second variation $\delta^2 H$ is positive or negative definite), then a displacement away from the solution would lead to a change in $H$, which is impossible; hence the solution must be stable to isovortical perturbations, thus yielding a sufficient condition for stability. Similarly, a necessary condition of instability is that the stationary point is a saddle [22]. The second variation $\delta^2 H$ can therefore be used, in principle, to assess stability; unfortunately, computing $\delta^2 H$ is often unfeasible, since solutions of practical interest are usually known only numerically. The implementation of Kelvin’s argument has thus been the subject of extensive debate.

Saffman and Szeto [21], having numerically found steady solutions for two co-rotating vortices, circumvented this difficulty as follows. Equation 1 can be interpreted as establishing extrema of $H$ in a velocity–impulse diagram (instead of an impulse–energy diagram). We deal with the second issue raised by Dritschel [4] by exploiting the fact that bifurcations are not structurally stable [19]; hence by introducing a small imperfection and re-computing the steady states, we obtain distinct solution branches, thus uncovering any bifurcations. All changes of stability are therefore apparent in an “Imperfect–Velocity–Impulse” (IVI) diagram.

In a forthcoming paper [13], we address both these issues by proposing a new approach to this problem. By building on ideas from dynamical systems theory, we show that changes of stability are associated with extrema in a velocity–impulse diagram (instead of an impulse–energy diagram). We deal with the second issue raised by Dritschel [4] by exploiting the fact that bifurcations are not structurally stable [19]; hence by introducing a small imperfection and re-computing the steady states, we obtain distinct solution branches, thus uncovering any bifurcations. All changes of stability are therefore apparent in an “Imperfect–Velocity–Impulse” (IVI) diagram.

Let us consider, as a schematic example of a typical construction of an IVI diagram, a possible scenario involving the detection of a subcritical bifurcation for a family of equilibrium solutions of the Euler equations (see Fig. 1). First, the steady base flows are computed, and the associated velocity–impulse diagram is plotted (Fig. 1a). Introducing a small imperfection in the governing equations, and re-computing the steady states, breaks the original curve into two distinct branches, revealing an extremum in $J$ (Fig. 1b). Since the extremum consists of a local minimum, it can be shown that stability is lost as the curve is traversed from right to left [13,12]. Therefore, if we suppose, for example, that the right-most portion of the branch is stable (marked by ‘S’ in the figure), the portion of the branch to the left of the minimum in $J$ will have one unstable mode (denoted by ‘1U’). Finally, by taking the strength of the imperfection to zero, we recover the underlying bifurcated solution branch (shown in red in Fig. 1c).

As an example of the practical use of the IVI diagram approach, we consider the stability of the family of elliptical vortices discovered by Kirchhoff [8]. While the elliptical family can be characterized analytically, detecting bifurcations through an IVI diagram requires finding equilibrium vortices numerically; the computational procedure is outlined in Sect. 2. In Sect. 3.1, we describe the unfolding of the IVI diagrams for the elliptical vortices. A large body of work exists regarding the stability of the Kirchhoff ellipses (see for example [10,16]); in Sect. 3.2, these results are employed to verify the efficacy of the IVI diagram approach.