Benoît Pier

Signalling problem in absolutely unstable systems

1 Introduction

In spatially homogenous systems, linear stability characteristics are derived from the dispersion relation \( \omega = \Omega(k) \) between the frequencies \( \omega \) and the wave numbers \( k \) of normal modes of the form \( e^{i(kx-\omega t)} \), with \( x \) and \( t \) denoting streamwise distance and time, respectively. These stability properties can be understood by resorting to different methods: temporal, spatial or spatio-temporal [8,10,18].

In a temporal approach, a spatially harmonic perturbation of real wave number \( k \) is considered. This wave-like initial perturbation evolves in time with a complex frequency \( \omega \). Its spatial structure, determined by the wave number \( k \), remains unchanged while its amplitude grows or decays with time. Growth or decay is determined by the sign of the temporal growth rate \( \omega_i \), while propagation takes place with a phase speed \( \omega_r/k \). This analysis based on real wave numbers and complex frequencies is known as the temporal problem.

In a spatial approach, localized harmonic forcing is applied with real frequency \( \omega \), say at \( x = 0 \). The spatial response to this forcing yields waves with complex wave numbers \( k \). The wavelength of the spatial response is determined by \( k_\ell \), the spatial growth or decay depends on \( k_\ell \); for \( x \to +\infty \), the spatial response grows when \( k_\ell < 0 \) and decays when \( k_\ell > 0 \); the reverse holds for \( x \to -\infty \). The analysis based on real frequencies and complex wave numbers is known as the spatial, or signalling, problem.

Throughout this paper, subscripts \( r \) and \( i \) denote real and imaginary parts of complex values.
The full spatio-temporal stability properties may be investigated by applying an impulsive localized perturbation: the analysis of the resulting wave packet yields the complete dispersion relation between complex wave numbers $k$ and complex frequencies $\omega$. While the impulsively started wave packet decays in stable media, a growing response develops from the impulse location in unstable systems. If the growing wave packet propagates away from its source and eventually leaves the medium unperturbed, the instability is said to be convective. If, by contrast, the instability grows in place and invades the system both upstream and downstream, the instability is said to be absolute. Convectively unstable (CU) systems do not display intrinsic dynamics and essentially behave as amplifiers: external perturbations are amplified while propagating through the system, and without continuous external input the medium returns to its unperturbed state. By contrast, absolutely unstable (AU) systems display non-trivial dynamics without external forcing: perturbations expand in both upstream and downstream directions so as to cover the entire domain and continue to grow exponentially at every point.

These stability concepts remain valid locally for spatially inhomogeneous systems, provided the characteristic inhomogeneity length scale is large compared to a typical instability length scale. However, the connection between local stability characteristics and the long-term global dynamics of spatially developing systems is far from obvious. In a linear framework, it has been shown [4,5,11] that the presence of local absolute instability is a necessary but not sufficient condition for global instability: in general an AU region of finite extent is required for a spatially developing medium to become globally unstable. Thus there exists a wide range of parameter settings for which a medium does not support any self-sustained fluctuations despite the presence of a region of local absolute instability. In such a situation, the linear signalling problem is legitimate and this is precisely the class of systems addressed in the present paper.

Globally stable but locally absolutely unstable systems are encountered in a variety of configurations of practical interest, among which wakes and boundary layers: the cylinder wake flow for Reynolds numbers in the range $25 < Re < 49$ [12], a class of “synthetic” wake flows [16,17], the wake of a sphere [15], the three-dimensional boundary layer produced by a rotating disk [7].

The paper is organized as follows: After formulating the problem in terms of the widely used partial differential complex Ginzburg–Landau equation (Sect. 2), its local (Sect. 3) and global (Sect. 4) stability properties are recalled. In Sect. 5, the correspondence between the complex space and frequency planes and the structure of the wave number branches are analysed. The complete analytic solution to the signalling problem is obtained in Sect. 6 in terms of asymptotic approximations and discussed in Sect. 7. These results are confirmed by direct numerical simulations in Sect. 8.

2 Problem formulation

Partial differential model equations account for the dynamics of a variety of physical systems [6] and are often tractable by analytical methods. The linearized complex Ginzburg–Landau model (1) has on many occasions proven to be a convenient testground to recognize and study generic features that have later been identified in a variety of situations. The same strategy is adopted here.

The system under consideration is assumed to be described by a complex scalar field $\psi(x,t)$ in an infinite one-dimensional spatially inhomogeneous domain and it is governed by

$$\frac{\partial \psi}{\partial t} = -i \left( \omega_0(X) + \frac{1}{2} \omega_{kk}(X) k_0(X)^2 \right) \psi + \omega_{kk}(X) k_0(X) \frac{\partial \psi}{\partial X} + \frac{i}{2} \omega_{kk}(X) \frac{\partial^2 \psi}{\partial X^2} + S(x,t),$$

where the complex functions $\omega_0(X)$, $k_0(X)$ and $\omega_{kk}(X)$ account for the local properties of the medium and only depend on a slow space variable $X = \epsilon x$. The coefficients of (1) have been cast in this form for reasons that will become clear in the next section. The weak inhomogeneity parameter $\epsilon \ll 1$ is defined as the ratio of the typical instability length scale to the inhomogeneity length scale of the medium. The source function $S(x,t)$ represents an externally applied forcing to be specified below. While Eq. 1 applies to the real $x$-axis, the functions $\omega_0(X)$, $k_0(X)$ and $\omega_{kk}(X)$ are assumed to be analytic and their continuation in the complex $X$-plane will be used in the following sections.

3 Local stability properties

In the subsequent discussion, constant use is made of the local properties of system (1). Local characteristics are derived from (1) by freezing $X$ to some arbitrary (possibly complex) value and studying the corresponding spatially homogenous system.