Three Inadequate Models

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Abstract. The connection between operational and denotational semantics is of longstanding interest in the study of programming languages. The emphasis has been on positive results, whether for adequacy or full abstraction. One normally considers the standard solution of an evident natural domain equation for the language; this is generally adequate but not fully abstract if one uses any of the usual categories of domains. One then tries other categories to get improved results.

Here we restrict ourselves to a standard category of domains and show, for an untyped $\lambda$-calculus with arithmetic, that inadequate models exist if one considers non-standard solutions to the domain equation. One model is inadequate, simpliciter; a second is adequate but inadequate when the language is extended by a “parallel or” construct; the third is adequate in the latter sense, but in it the $Y$-combinator does not denote the least fixed point operator.

We also consider whether it is possible to do better than the standard solution as regards full abstraction. Surprisingly this question only makes sense for solutions which are adequate for the extended language. For these the standard solution is indeed closest to full abstraction, justifying the use of non-standard categories.

Keywords: Adequacy; Counterexample; Full abstraction; PCF

1. Introduction

The connection between operational and denotational semantics is of long-standing interest in the study of programming languages. One naturally seeks positive results. For example in [FiP94, Sim99] adequacy results are given for models in a variety of categories. Again, the failure of full abstraction in the standard models constructed using complete partial orders and continuous functions [Plo77, Mil77] prompted the exploration of other categories (see, for example, [BCL85, FJM96, AbM98, AmC98]) with varying degrees of success.

In this paper we interest ourselves in counterexamples in order to make a case that these natural avenues of research had a degree of necessity. To this end, we construct inadequate models and investigate whether one can do better than the standard model, but still stay in the category of complete partial orders. (In contrast, an inadequate standard model of PCF is given in [Sim99] – but in a specially constructed category.)

We consider just one example, an untyped call-by-name $\lambda$-calculus $\mathcal{L}$, whose expressions $M$ are given by

\[
M ::= x \mid \emptyset \mid \text{succ}(M) \mid \lambda x.M \mid \delta(M, M, M) \mid \text{pred}(M) \mid \text{if } M = 0 \text{ then } M \text{ else } M \mid MM
\]

where $x$ runs over a fixed countably infinite set of variables. This is, essentially, the language considered by Pitts in [Pit93], but with the trivial variation of using natural numbers rather than integers (as will be
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seen from the operational semantics) and with the more significant addition of $\beta$ (to discriminate between functions and natural numbers).

There is a natural domain equation associated to this language:

$$D \equiv (D \Rightarrow D) + N$$

and its standard solution is adequate. We give a non-standard solution to the domain equation which is not adequate, confirming that some restriction in the class of models is needed for adequacy. The idea of the construction dates back to Park [Par76], who gave a non-standard model of the pure untyped $\lambda$-calculus in which the paradoxical combinator does not denote the least fixed-point operator.

One would hope next to show that the standard model, even though itself not fully abstract, gets as close as possible to that ideal among the models available or, if not, at least among the adequate models. The position here is not quite what one might expect. It turns out that if, in a suitable sense, one model is less abstract than another, then they make the same termination predictions for $L^+$, an extension of $L$ by a ‘parallel or’ construct. It then follows that a model is comparably abstract with the standard one if it is $L^+$-adequate. This is somewhat surprising as one might, for example, have expected that the equational theory of any model adequate for $L^+$ would be included in that of the standard model, rather than the prima facie stronger requirement of adequacy for $L^+$. We are able to give a non-standard solution of the domain equation which is adequate for $L$ but not for $L^+$ and so consideration of the stronger adequacy condition really is required; this is our technically most involved result.

So far the standard model is the only example we have that is adequate for $L^+$, but it is not difficult to find other solutions of the domain equation which are also adequate for $L^+$, and in fact we can even find one in which the Y-combinator does not denote the least fixed-point operator. This model is inadequate in the sense that no analogue of the Hyland-Wadsworth approximation theorem [Bar84] can hold.

Section 2 below presents various technical preliminaries. The three inadequate models are presented in Sections 3, 5 and 6, with the ‘less abstract than’ relation being considered in Section 4; Section 7 is a discussion section, placing our work in a broader context. The Appendix presents two results on full abstraction for the standard model; while perhaps not quite folklore, they are hardly unexpected: that the model is not fully abstract for $L$, but is for $L^+$. While we do define the particular flavour of domain used (cppos) the paper is not completely self-contained, and appropriate knowledge of domain theory is assumed.

2. Technical Preliminaries

2.1. Syntax

We have given the syntax of $L$ above. Free and bound variables are defined as usual ($\lambda$ is the only binding operator), as is simultaneous substitution $M[N_1/x_1, \ldots, N_n/x_n]$; we write $L^0$ for the set of closed terms of $L$.

The operational semantics of $L$ is given by the rules in Fig. 1, giving an inductive definition of an evaluation relation $M \Rightarrow V$ between closed terms $M$ and (syntactic) values $V$; the latter are taken to be those closed terms which are either abstractions $\lambda x:N$ or numerals $n = \text{succ}(0)$. We define the termination property $M \Downarrow^L$ for $L$-terms $M$ to be that $M \Rightarrow V$, for some $V$.

The language $L^+$ is obtained from $L$ by adding a ‘parallel or’ construct $\text{por}(M, N)$; its values are again closed abstractions and numerals. The evaluation rules are as before, together with

\[
\begin{align*}
M \Rightarrow 0 & \quad \Rightarrow 0 \\
\text{por}(M, N) \Rightarrow 0 \\
M \Rightarrow 1 & \\
\text{por}(M, N) \Rightarrow 1 \\
N \Rightarrow 1 & \\
\text{por}(M, N) \Rightarrow 1
\end{align*}
\]

yielding an evaluation relation $M \Rightarrow^L V$ for $L^+$ (and a termination predicate $M \Downarrow^{L^+}$).

There is a natural equational theory of ‘$\beta$-rules’ for $L$ with axioms:

1. $\hat{\beta}(\lambda x.L, M, N) = M$
2. $\hat{\beta}(n, M, N) = N$