The evaluation of two-sided orthant probabilities for a quadrivariate normal distribution

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Abstract In this paper it is shown how a general two-sided orthant probability for a quadrivariate normal distribution can be evaluated by a one-dimensional numerical integral calculation. The quadrivariate normal distribution can have any covariance matrix and any mean vector. This affords a practical and efficient method for the calculation of these probabilities which is superior to simulation methods. The implementation of the algorithm is discussed, and some examples of its performance are provided.

Keywords Multivariate normal distribution · Quadrivariate normal distribution · Numerical integration · Computational intensity · Orthant probability

1 Introduction

Let the random variables $X = (X_1, X_2, X_3, X_4)$ have a quadrivariate normal distribution with mean vector $\mu$ and positive-definite covariance matrix $\Sigma$. This article addresses the evaluation of the two-sided orthant probability

$$P(l_i \leq X_i \leq u_i; 1 \leq i \leq 4)$$

where some of the $l_i$ may be equal to $-\infty$ and some of the $u_i$ may be equal to $\infty$. With a suitable linear transformation of the random variables $X$ any probability of this kind can be expressed as a two-sided orthant probability for random variables with
a mean $\mu = 0$ and with $\Sigma$ equal to a correlation matrix with all diagonal elements equal to one. In this paper it is shown how (1) can be evaluated numerically with a one-dimensional integral calculation.

The evaluation of (1) has been recognized as an important and interesting problem which has received considerable attention in the statistical literature. However, attention has primarily focused on centered one-sided orthant probabilities where the $l_i$ are all zero, the $u_i$ are all $\infty$, and $\mu = 0$, as in McFadden (1960) and Sondhi (1961) where series expansions and approximations for this case are discussed. In addition, Cheng (1969) considers the evaluation of centered one-sided orthant probabilities for certain specific forms of the covariance matrix $\Sigma$.

Abrahamson (1964) and Gehrlein (1979) show how a centered one-sided orthant probability can be evaluated as a series of terms which can each be evaluated as a one-dimensional integral, and Miwa et al. (2003) show that this can be done for a one-sided orthant probability with any mean $\mu$. The general two-sided orthant probability (1) can be expressed as a linear combination of one-sided orthant probabilities, and then the algorithm of Miwa et al. (2003) can be used to evaluate each one-sided orthant probability as a series of one-dimensional integral calculations. However, the explicit equation presented in this paper provides a simpler and more efficient procedure. The methodology utilized in this paper is a special case of a more general recursive integration methodology. Recursive relationships have been used in many areas of probability and statistics and it has been recognized that they can provide a very useful analysis tool, as discussed in Hayter (2006).

Considerable research has been done on the related problem of calculations for multivariate $t$ distributions, particularly when the inequalities of interest involve contrasts of the means. See, for example, Bretz et al. (2001), Genz and Bretz (2002), Somerville (1998) and Uusipaikka (1985). Notably, Genz and Bretz (2009) provides an excellent review on the computation of multivariate normal and $t$ probabilities. However, for the problem considered in this paper, most of the methods proposed in the literature require some element of simulation, and the objective of this paper is to illustrate the usefulness of a one-dimensional numerical integration technique.

A two-sided orthant probability for a bivariate normal distribution can be expressed in terms of normal random variables $X_1$ and $X_2$ which have zero means, variances of one, and a correlation $\rho$, with

$$P(l_i \leq X_i \leq u_i; 1 \leq i \leq 2) = \int_{x=l_1}^{u_1} \phi(x) \left[ \Phi \left( \frac{u_2 - \rho x}{\sqrt{1 - \rho^2}} \right) - \Phi \left( \frac{l_2 - \rho x}{\sqrt{1 - \rho^2}} \right) \right] dx.$$

(2)

A two-sided orthant probability for a trivariate normal distribution can be expressed in terms of normal random variables $X_1$, $X_2$ and $X_3$ which have zero means, variances of one, and correlations $\rho_{ij}$, $1 \leq i < j \leq 3$. It is shown in Hayter (2011) that a trivariate two-sided orthant probability either factors into univariate and bivariate two-sided orthant probabilities, or it can be expressed as