Uncoupled automata and pure Nash equilibria

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Abstract We study the problem of reaching a pure Nash equilibrium in multi-person games that are repeatedly played, under the assumption of uncoupledness: EVERY player knows only his own payoff function. We consider strategies that can be implemented by finite-state automata, and characterize the minimal number of states needed in order to guarantee that a pure Nash equilibrium is reached in every game where such an equilibrium exists.

Keywords Automaton · Nash equilibrium · Uncoupledness

1 Introduction

We study the problem of reaching Nash equilibria in multi-person games, where the players play the same game repeatedly. The main assumption, called uncoupledness (see Hart and Mas-Colell 2003), is that every player knows only his own utility function. The resulting play of the game yields an uncoupled dynamic. Hart and Mas-Colell show in 2003 that if the game is played in continuous time, and the moves of every player are deterministic, then uncoupled dynamics cannot always lead to Nash equilibria. In Hart and Mas-Colell (2006) they show that the situation is different when stochastic moves are allowed and the game is played in discrete time: if the players know the history of play, then there are uncoupled strategies that lead to a Nash equilibrium. The question is whether it is necessary to know the whole history in order to reach a Nash equilibrium. The answer is no. It was proved in Hart

1 I.e., the past actions of all the players.

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and Mas-Colell (2006), Theorems 4 and 5, that under the assumption of uncoupled-ness, convergence of the long-run empirical distribution of play to a (pure or mixed) Nash equilibrium can be guaranteed by using only the history of the last \( R \) periods of play, for some finite \( R \). This is called a \textit{finite-recall strategy}. Although finite-recall uncoupled strategies can guarantee convergence of the distribution of play to a Nash equilibrium, it is shown in Hart and Mas-Colell (2006), Theorem 6, that this \textit{cannot} hold for the period-by-period behavior probabilities. If however, instead of finite recall one uses \textit{finite memory} (e.g., finitely many periods of history but not necessarily the last ones), then the convergence of the behavior can be guaranteed as well (Hart and Mas-Colell 2006, Theorem 7).

This leads us to the study of uncoupled strategies with finite memory, i.e., \textit{finite-state automata}. In this paper, we deal with convergence to pure Nash equilibria in games which have such equilibria. In Hart and Mas-Colell (2006), Theorem 3, it is shown that in order to guarantee convergence to pure Nash equilibria one needs recall of size \( R = 2 \). Since finite recall is a special case of finite automata, the question we address here concerns the \textit{minimum number of states required} for uncoupled finite automata to reach a pure Nash equilibrium. There are four classes of finite-state automata: the actions in every state can be deterministic (pure) or stochastic (mixed), and the transitions between states can be deterministic or stochastic. We will analyze each of the four classes in turn.

Section 2 presents the model, defines the relevant concepts and present the total results of the paper. Since the results are different for two-player games than for games with more than two players, we consider two-player games in Sect. 3 and \( n \)-player games for \( n \geq 3 \) in Sect. 4. In Sects. 3 and 4, we discuss each of the four automata classes separately. Appendix A and Appendix B containing the proofs of Theorems 6 and 7.

2 The model

2.1 The game

A basic static (one-shot) game \( \Gamma \) is given in strategic (or normal) form as follows. There are \( n \geq 2 \) players, denoted \( i = 1, 2, \ldots, n \). Each player \( i \) has a finite set of pure actions \( A^i = \{a^i_1, \ldots, a^i_{m_i}\} \); let \( A := A^1 \times A^2 \times \cdots \times A^n \) be the set of action combinations. The payoff function (or utility function) of player \( i \) is a real-valued function \( u^i : A \to \mathbb{R} \). The set of mixed (or randomized) actions of player \( i \) is the probability simplex over \( A^i \), i.e., \( \Delta(A^i) = \{x^i = (x^i(a^i_j))_{j=1,\ldots,m^i} : \sum_{j=1}^{m^i} x^i(a^i_j) = 1 \text{ and } x^i(a^i_j) \geq 0 \text{ for } j = 1, \ldots, m^i\} \); payoff functions \( u^i \) are multilinearly extended, and so \( u^i : \Delta(A^1) \times \Delta(A^2) \times \cdots \times \Delta(A^n) \to \mathbb{R} \).

We fix the set of players \( n \) and the action sets \( A^i \), and identify a game by its \( n \)-tuple of payoff functions \( U = (u^1, u^2, \ldots, u^n) \). Let \( \mathcal{U}^i \) be the set of payoff functions of player \( i \), and \( \mathcal{U} := \mathcal{U}^1 \times \cdots \times \mathcal{U}^n \).

Denote the actions of all the players except player \( i \) by \( a^{-i} \), i.e., \( a^{-i} = (a^1_{-i}, \ldots, a^{i-1}_{-i}, a^{i+1}_{-i}, \ldots, a^n_{-i}) \), and denote the set of actions of all the players except player \( i \) by \( A^{-i} = A^1 \times \cdots \times A^{i-1} \times A^{i+1} \times \cdots \times A^n \). An action \( a^i_j \in A^i \) will be called a best