Evolutionarily stable sets

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Abstract. This paper provides definitions for the evolutionary stability of sets of strategies based on simple fitness comparisons in the spirit of the definition of an evolutionarily stable strategy (ESS) by Taylor and Jonker (1978). It compares these with the set-valued notions of Thomas (1985d) and Swinkels (1992). Provided only that the fitness function is analytic, our approach yields an alternative characterization of Thomas’ evolutionarily stable sets (ES Sets) which does not rely on the structure or topology of the underlying strategy space. Moreover, these sets are shown to have a very special geometric structure and to be finite in number. For bimatrix games ES Sets are shown to be more uniformly robust against mutations than apparent from the definition and hence to be equilibrium evolutionarily stable sets in the sense of Swinkels (1992).

Key words: evolutionary stability, equilibrium components

1. Introduction

The notion of an ESS due to Maynard Smith and Price (1973) is a central concept in evolutionary game theory. The notion led to important insights for many simple models of evolutionary conflicts. However, in more complex models, as in models with sexual reproduction or in models based on extensive games, one faces the difficulty that ESS often do not exist for a very basic reason. Namely, in more complex models most strategies have “twins” which behave similar in every evolutionary relevant aspect. The requirement that

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an ESS has to have a fitness advantage over all strategies other than itself can then not be satisfied with respect to these “twins”. One can overcome this difficulty by considering sets of “evolutionarily similarly robust” strategies which have the required fitness advantages only with respect to all strategies outside the set. Although one does not obtain existence for every game in this way the range of applicability of evolutionary stability concepts is considerably widened.\footnote{Applications are discussed in Thomas (1985b), Thomas (1985c), Cressman (1992), Balkenborg (1995), Schlag (1994b) and Schlag (1994a).}

This idea has been formalized in the notion of an \textit{evolutionarily stable set (ES Set)} by Thomas (1985d). However, Thomas gives little motivation for his formalization and we are not at ease with the use of the structure and topology of the mixed strategy space in his definition. It seems to rely on an identification of mixed strategies and population states that may not always be justified. Therefore we offer alternative, hopefully more intuitive, formalization which are directly based on Taylor and Jonker’s classical notion of an ESS and which avoid any direct use of the structure of the mixed strategy space.

Taylor and Jonker’s notion has been criticized early on by Vickers and Cannings (1987) because it allows the size of a mutation against which an ESS has to be robust to vary in dependence on the type of mutation. Consequently, the role of mutation sizes will be a central theme in our paper.

To give a better account of our approach it is useful to introduce the following terminology which will be used throughout. Consider a “bimorphic” scenario where initially all members of a population use the same strategy $p$ and where then a small fraction of the population mutates and switches to a different strategy $q$. For $\varepsilon > 0$ we say that $q$ is $\varepsilon$-\textit{driven out} by $p$ if $p$ has a higher fitness than $q$ in the post entry population whenever the fraction of mutant players does not exceed $\varepsilon$. If, conversely, $q$ has a higher fitness than $p$ in the post-entry population for all sufficiently small such mutations we say that $q$ \textit{spreads given} $p$.

In this terminology an ESS as defined by Taylor and Jonker is a strategy $p$ which can $\varepsilon$-\textit{drive} out every other strategy $q \neq p$. Correspondingly, we call a set of strategies a simple evolutionarily stable set \textit{(simple ES Set)} if a) no strategy in the set can spread given any other strategy in the set in the set and if b) every strategy in the set can $\varepsilon$-\textit{drive} out every other strategy not in the set for some $\varepsilon > 0$.

Hereby property a) is meant to be a weak requirement of the type that all strategies in the set should be equally “evolutionarily robust” while property b) requires that each strategy in the set must have the same property as an ESS with respect to “outsiders”.

Note that the structure of the strategy space and its topology plays no role in our definition of a simple ES Set. Our definitions do not rely on the convexity of the strategy space. We do not identify a scenario where a fraction $1-x$ of the population plays $p$ and a fraction $x$ plays $q$ with a scenario where all members of the population play the mixed strategy $(1-x)p+xq$, as it seems to be implicit in Thomas’ definition of an ES Set. Also, we do not require a simple ES Set to be closed, again in contrast to Thomas’ definition. All we need for our concepts to be well-defined is that we know for each strategy pair $p,q$ the fitness these strategies would have in a “bimorphic” population