Stay-in-a-set games*

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Abstract. There exists a Nash equilibrium (ε-Nash equilibrium) for every n-
person stochastic game with a finite (countable) state space and finite action
sets for the players if the payoff to each player i is one when the process of
states remains in a given set of states Gi and is zero otherwise.

Key words: Stochastic game, Nash equilibrium, gambling theory, games of
survival.

1. Introduction

Consider an n-person stochastic game with a countable, nonempty set of states
S, n players 1, 2, ... , n, finite action sets A1, A2, ... , An for the players, a con-
tditional probability distribution q on S given S × (A1 × A2 × ... × An) called
the law of motion, and bounded, real valued payoff functions f1, f2, ... , fn defined on the history space
H = S × A × S × A × ... , where A = A1 × A2 × ... × An.

Play begins at an initial state x0 = x ∈ S. Each player i independently se-
lects a mixed action a1 i with a probability distribution σi(x) belonging to
İ(Ai), the set of probability measures on Ai. Given x0 and the chosen actions
a1 = (a1 1, a1 2, ... , a1 n) ∈ A, the next state x1 has distribution q(·|x0, a1). Then
again each player i independently selects a2 i with a distribution σi(x0, a1 , x1) and,
given a2 = (a2 1, a2 2, ... , a2 n), the next state x2 has distribution q(·|x1, a2).

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Play continues in this fashion thereby generating a random history $h = (x_0, a^1, x_1, a^2, \ldots) \in H$.

A function $\sigma$, that specifies for each partial history $p = (x_0, a^1, x_1, a^2, \ldots, x_n)$ the conditional distribution $\sigma_i(p) \in \mathcal{P}(A_i)$ for player $i$’s next action $a^{n+1}$ is called a strategy for player $i$. A profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ consists of a strategy $\sigma_i$ for each player $i$. A set $x$ and a profile $\sigma$ together with the law of motion $q$ determine a probability distribution $P_{x, \sigma}$ on the history space $H$, which we equip with the product sigma-field when the sets $S$ and $A$ are each given their sigma-field of all subsets. We write $E_{x, \sigma}$ for the expectation operator associated with $P_{x, \sigma}$.

Assume now that the payoff functions $\phi_i : H \rightarrow \mathbb{R}$ are bounded and measurable. If the initial state of the game is $x$ and each player $i$ chooses a strategy $\sigma_i$, then the return to each player $i$ is the expectation $E_{x,\sigma_i} \phi_i$, where $\sigma$ is the profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$.

For $\varepsilon > 0$, an $\varepsilon$-equilibrium at the initial state $x$ is a profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ such that, for all $i = 1, 2, \ldots, n$,

$$E_{x,\sigma} \phi_i \geq \sup_{\mu_i} E_{x,(\sigma_1, \ldots, \sigma_{i-1}, \mu_i, \sigma_{i+1}, \ldots, \sigma_n)} \phi_i - \varepsilon,$$

where $\mu_i$ ranges over the set of all strategies for player $i$. In other words, each $\sigma_i$ guarantees an expected payoff to player $i$ which is within $\varepsilon$ of the best possible expected payoff for player $i$ when every other player $j \neq i$ plays $\sigma_j$. A 0-equilibrium is called a Nash equilibrium.

Let $r_i : S \times A \rightarrow \mathbb{R}$ be a daily reward function for player $i, i = 1, 2, \ldots$. It has long been known that Nash equilibria exist for some interesting payoff functions such as a discounted payoff

$$\phi_i(h) = \sum_{n=0}^{\infty} \beta^n r_i(x_n, a^{n+1}), \quad 0 < \beta < 1,$$

(cf. Mertens and Parthasarathy (1991) and the references given there), but need not exist for other payoff functions such as an average reward

$$\phi_i(h) = \limsup_n \frac{1}{n} \sum_{k=0}^{n-1} r_i(x_k, a^{k+1})$$

even for a two-person, zero-sum game with a finite state space (cf. Gillette (1957), Blackwell and Ferguson (1968) for a famous counterexample and Vieille (2000a, 2000b) for the existence of “equilibrium payoffs” in two-person, non-zero sum games.)

To define the payoff functions we study here, let $G_1, G_2, \ldots, G_n$ be subsets of the state space $S$. Then let $G_1^\circ, G_2^\circ, \ldots, G_n^\circ$ be the subsets of $H$ defined by

$$G_i^\circ = \{ h = (x_0, a^1, x_1, a^2, \ldots) : x_k \in G_i \text{ for all } k = 0, 1, \ldots \},$$

and take the payoff function $\phi_i$ to be the indicator function of $G_i^\circ$ for $i = 1, 2, \ldots, n$. Thus each player $i$ receives a payoff of 1 if the process of states $x_0, x_1, \ldots$ stays in $G_i$ and receives nothing otherwise. A stochastic game with payoff functions of this form we call a stay-in-a-set game.

### 1.1 Theorem

An $n$-person stay-in-a-set game with a finite state space has a Nash equilibrium at every initial state.