A new non-linear $AR(1)$ time series model having approximate beta marginals

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Abstract We consider the mixed $AR(1)$ time series model

\[
X_t = \begin{cases} 
\alpha X_{t-1} + \xi_t & \text{w.p. } \frac{\alpha^p}{\alpha^p - \beta^p}, \\
\beta X_{t-1} + \xi_t & \text{w.p. } -\frac{\beta^p}{\alpha^p - \beta^p}
\end{cases}
\]

for $-1 < \beta^p \leq 0 \leq \alpha^p < 1$ and $\alpha^p - \beta^p > 0$ when $X_t$ has the two-parameter beta distribution $B_2(p, q)$ with parameters $q > 1$ and $p \in \mathcal{P}(u, v)$, where

\[
\mathcal{P}(u, v) = \{u/v : u < v, \ u, \ v \ \text{odd positive integers}\}.
\]

Special attention is given to the case $p = 1$. Using Laplace transform and suitable approximation procedures, we prove that the distribution of innovation sequence for $p = 1$ can be approximated by the uniform discrete distribution and that for $p \in \mathcal{P}(u, v)$ can be approximated by a continuous distribution. We also consider estimation issues of the model.

Keywords Approximate beta marginal · Beta distribution · First order autoregressive model · Kummer function of the first kind

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1 Introduction

The development of time series models with non-Gaussian marginals began in the early eighties of the last century. For comprehensive accounts of the developments, we refer the readers to Popović (2010) and Popović et al. (2010).

If the Laplace transform of the considered marginals takes an elementary form then it possible to determine the exact distribution of the corresponding innovation sequence in closed form. If it is not the case (for example, beta marginals) then it is not possible to determine the exact distribution of the corresponding innovation sequence in closed form.

Consider the two-parameter beta distribution $B_2(p, q)$. The corresponding Laplace transform involves a special function, the Kummer function of the first kind. For elementary forms, it is necessary some approximations are made. If the parameters, $p$ and $q$, are taken to be large then elementary approximations can be obtained. The resulting distribution determined so is known as the approximate beta distribution. The approximate beta can be used to derive the distribution of the corresponding innovation sequence.

This process of approximating the distribution of innovation sequences is very helpful for those distributions not having elementary Laplace transforms. This process was introduced for the first time by Popović (2010) and Popović et al. (2010).

The first nonlinear model with an approximate beta distribution, $AB_{p, q}$, was considered in Popović et al. (2010). The first linear model with an $AB_{p, q}$ was considered in Popović (2010). In both cases, it was shown that the distribution of innovation sequences can be approximated by a uniform discrete distribution ($p = 1$) or by a continuous distribution ($p \in (0, 1)$).

In this note, we consider a new mixed autoregressive first order time series model based on the two-parameter beta distribution $B_2(p, q)$ with parameters $q > 1$ and $p \in P(u, v)$, where

$$P(u, v) = \{u/v : u, v \text{ odd positive integers}\} \subset (0, 1).$$

We derive the distribution of the corresponding innovation sequence as in Popović (2010) and Popović et al. (2010). We examine the cases $p = 1$ and $p \in (0, 1)$ separately. We denote the time series model corresponding to $AB_{p, q}$ by $ARAB_1$. For the particular case $p = 1$, we use the notation $ARAB^*(1)$.

The newly introduced model takes the mixed form $X_t = \Lambda X_{t-1} + \xi_t$, where $X_t$ are beta distributed and $\Lambda$ is a random variable taking two possible values. Time series models of this form have been studied by many authors. For example, McKenzie (1985, p. 990) considers $X_t = \Lambda X_{t-1} + \xi_t$ with $X_t$ beta distributed. Pötzelberger (1990, p. 177) considers $X_t = \Lambda X_{t-1} + \xi_t$ with $X_t$ having the uniform distribution. Hwang and Basawa (2005) consider $X_t = \Lambda X_{t-1} + \xi_t$ with $\Lambda$ taking the values

$$\Lambda = \begin{cases} \theta & \text{w.p. } \alpha, \\ -\theta & \text{w.p. } 1 - \alpha \end{cases}$$ (1.1)