On maximum likelihood prediction based on Type II doubly censored exponential data

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Abstract. In this paper, the maximum likelihood predictor (MLP) of the $k$th ordered observation, $t_k$, in a sample of size $n$ from a two-parameter exponential distribution as well as the predictive maximum likelihood estimators (PMLE’s) of the location and scale parameters, $\theta$ and $\beta$, based on the observed values $t_r, \ldots, t_s$ ($1 \leq r \leq s < k \leq n$), are obtained in closed forms, contrary to the belief they cannot be so expressed. When $\theta$ is known, however, the PMLE of $\beta$ and MLP of $t_k$ do not admit explicit expressions. It is shown here that they exist and are unique; sharp lower and upper bounds are also provided. The derived predictors and estimators are reasonable and also have good asymptotic properties. As applications, the total duration time in a life test and the failure time of a $k$-out-of-$n$ system may be predicted. Finally, an illustrative example is included.

Key words: Predictive likelihood, Type II double censoring, two-parameter exponential distribution, order statistics

1 Introduction

Predictive problems in statistics are those in which the unknown quantities of interest are future (or missing) random variables. A typical situation involves the prediction of the behaviour of future observations based on earlier ones (see Ogunyemi and Nelson, 1997, and references therein). Several alternative approaches to this problem have generally been used, according to the model assumed and the type of information available.

The two-parameter exponential distribution, $\text{Exp}(\theta, \beta)$, provides a population model which is useful in several areas of statistics. In survival and reliability analysis, this distribution plays an important role and is often used to describe certain lifetime data in the biomedical area, as well as component
failure observations or the reliability of equipment in industrial applications. The probability density function of an \( \text{Exp}(\theta, \beta) \) variate is given by

\[
f(t; \theta, \beta) = \beta^{-1} \exp\left\{-\frac{t - \theta}{\beta}\right\}, \quad t > \theta, \quad \beta > 0,
\]

where \( \theta \) and \( \beta \) are the respective location and scale parameters. Its cumulative distribution function is

\[
F(t; \theta, \beta) = 1 - \exp\left\{-\frac{t - \theta}{\beta}\right\}, \quad t > \theta.
\]

References on this model may be found, among many others, in Bain (1978) and Lawless (1982). In lifetime data analysis, the location parameter, \( \theta \), may be interpreted as an unknown point at which “life” begins or as a “guarantee time” during which failure cannot occur. The \( \text{Exp}(\theta, \beta) \) model arises as the limiting form of the distribution of a minimum of random samples from some densities with support on \( (\theta, \infty) \). This property is often a justification for its use in reliability studies in which a complex mechanism fails when any one of its many components fails.

In reliability studies, due to time limitations and/or other restrictions on data collection, several lifetimes of units put on test may not be observed. In addition, sometimes the lowest and/or highest few observations in a sample could be due to some negligence or some other extraordinary reasons. In such cases, it is convenient to remove those outlying observations. Type II censored samples are considered here, whereby, in an ordered sample of size \( n \), a known number of observations is missing at either end (single censoring) or at both ends (double censoring). Doubly censored samples have been considered, among other authors, by Sarhan (1955), Harter and Moore (1968), Kambo (1978), Bhattacharyya (1985), Tiku et al. (1986), Leemis and Shih (1989), Raqab (1995), Balakrishnan and Chan (1995), and Lalitha and Mishra (1996).

In order to make predictions on future (or missing) observations, Kaminisky and Rhodin (1985) present an approach based on the maximum likelihood (ML) principle. That approach is adopted in this paper to derive predictors of future exponential order statistics based on a Type II doubly censored sample.

The problem of finding the MLP of the \( k \)th ordered observation, \( t_k \), in a sample of size \( n \) from a two-parameter exponential population as well as the PMLE’s of the unknown parameters, based on the observed values \( t_r, \ldots, t_s \) \( (1 \leq r \leq s < k \leq n) \), is discussed in Section 2. The ML principle provides a flexible procedure for predicting \( t_k \) when the observed sample is Type II doubly censored. If one considers a system composed by \( n \) independent and identical components with exponential lives and such that the failure of the system occurs as soon as \( k \) of the components fail (this is called a \( k \)-out-of-\( n \) system), the lifetime of the system is that of the \( k \)th order statistic. Thus, an application of the present paper is to predict the failure time of a \( k \)-out-of-\( n \) system of independent exponentially distributed components. Another is to predict the total duration time in a life test on the basis of early results in the test. Finally, as a numerical illustration, an example is presented in Section 3.

### 2 Maximum likelihood prediction

Consider a random sample of size \( n \) from an \( \text{Exp}(\theta, \beta) \) distribution and let \( t_r, \ldots, t_s \) be the ordered observations remaining when the \( (r - 1) = np \)