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Properties of game options

Received: May 2004 / Revised version: May 2005 / Published online: 10 November 2005
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Abstract A game option is an American option with the added feature that not only the option holder, but also the option writer, can exercise the option at any time. We characterize the value of a perpetual game option in terms of excessive functions, and we use the connection between excessive functions and concave functions to explicitly determine the value in some examples. Moreover, a condition on the two contract functions is provided under which the value is convex in the underlying diffusion value in the continuation region and increasing in the diffusion coefficient.

Keywords Optimal stopping games · Game options · Excessive functions · Volatility · Price orderings

Mathematics Subject Classification (2000) Primary 91A15 · Secondary 60G40 · 91B28

1 Introduction

Consider a diffusion $X$ with state space $I \subseteq \mathbb{R}$, where $I$ is an interval with endpoints $a, b$ such that $-\infty \leq a < b \leq \infty$. Assume that $X$ is a stochastic process with dynamics

$$dX_t = \mu(X_t) \, dt + \sigma(X_t) \, dB_t, \quad X_0 = x,$$

where $B$ is a standard Brownian motion and $\mu(\cdot)$ and $\sigma(\cdot) > 0$ are continuous functions. We assume that $X$ is regular in the interior of $I$, i.e., $X_t$ reaches $y$ in finite time with positive probability for every starting point $x \in (a, b)$ and every point...
If a boundary point of $I$ can be reached in finite time, then we assume that it is absorbing. Denote by $\overline{T}$ the union of the set of such regular boundary points and the interval $I$ (with this terminology, $\overline{T}$ is not necessarily a closed interval). Let $g_1, g_2 : \overline{T} \to \mathbb{R}$ be two continuous functions such that $0 \leq g_1 \leq g_2$. We consider the following continuous time Dynkin game between two players, the buyer and the seller: each player chooses a stopping time, say $\tau$ and $\gamma$, respectively. Then, at the time $\tau \wedge \gamma := \min\{\tau, \gamma\}$, the seller pays the amount

$$g_1(X_{\tau})1_{\{\tau \leq \gamma\}} + g_2(X_{\gamma})1_{\{\gamma < \tau\}}$$

to the buyer. Given a constant discounting rate $\beta \geq 0$, we define the value of this game as

$$V(x) := \sup_{\tau} \inf_{\gamma} E_x R(\tau, \gamma)$$

(2)

for $x \in \overline{T}$, where

$$R(\tau, \gamma) := e^{-\beta \tau} g_1(X_{\tau})1_{\{\tau \leq \gamma\}} + e^{\beta \gamma} g_2(X_{\gamma})1_{\{\gamma < \tau\}}$$

and the index $x$ of the expected value indicates that the diffusion is started at $x$ at time 0. Moreover, the supremum and the infimum are taken over random times $\tau$ and $\gamma$ that are stopping times with respect to the filtration generated by the Brownian motion $B$, and we use the convention that $g_i(X_{\sigma}) = 0$ on $\{\sigma = \infty\}$. Note that the value $V$ satisfies

$$g_1(x) \leq V(x) \leq g_2(x)$$

for all $x \in \overline{T}$ (choose $\tau = 0$ and $\gamma = 0$, respectively). We assume throughout this article that for any starting point $x$ the stopping times

$$\tau^* := \inf\{t; V(X_t) = g_1(X_t)\}$$

and

$$\gamma^* := \inf\{t; V(X_t) = g_2(X_t)\}$$

form a saddle point, i.e., they satisfy

$$V(x) = \sup_{\tau} E_x R(\tau, \gamma^*) = \inf_{\gamma} E_x R(\tau^*, \gamma) = E_x R(\tau^*, \gamma^*).$$

It follows that $V(\cdot)$ is continuous and that the order of the supremum and the infimum in (2) is not essential. A sufficient condition to ensure that $(\tau^*, \gamma^*)$ is a saddle point is that $\beta > 0$ and $g_2$ is bounded (Lepeltier and Maingueneau 1984).

It is well-known that the value of an optimal stopping problem (formally corresponding to $g_2 \equiv \infty$) can be characterized as the smallest $\beta$-excessive majorant of the contract function $g_1$, compare Dynkin (1963) and Fakeev (1971). Dynkin (1965) and Dayanik and Karatzas (2003) show that excessivity of a function is equivalent to concavity of the function in a certain generalized sense. This leads to a characterization of the value of optimal stopping problems as the smallest concave majorant of the contract function $g_1$. Moreover, since the smallest concave majorant of a function is easily found by inspection, this characterization leads