Price systems constructed by optimal dynamic portfolios

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Abstract. The paper studies connections between arbitrage and utility maximization in a discrete-time financial market. The market is incomplete. Thus one has several choices of equivalent martingale measures to price contingent claims. Davis determines a unique price for a contingent claim which is based on an optimal dynamic portfolio by use of a ‘marginal rate of substitution’ argument. Here conditions will be given such that this price is determined by a martingale measure and thus by a consistent price system. The underlying utility function $U$ is defined on the positive half-line. Then dynamic portfolios are admissible if the terminal wealth is positive. In case of the logarithmic utility function, the optimal dynamic portfolio is the numeraire portfolio.

Key words: martingale measure, utility maximization, optimal dynamic portfolio, pricing of options, dynamic programming


1 Introduction and summary

In discrete time $t = 0, 1, \ldots, T$, a frictionless financial market is studied which is free of arbitrage opportunities but incomplete. It is known (see Harrison & Pliska 1981, Jacod & Shiryaev 1998) that in discrete time the assumption of completeness is a severe restriction and in general one has several choices of equivalent martingale measures (from a convex set). It is further known (see Harrison & Kreps 1979) that each martingale measure corresponds to a consistent price system. Thus in incomplete markets, no preference independent pricing of contingent claims is possible.

In the market $1 + d$ assets can be traded. One of them with price process $\{B_t, 0 \leq t \leq T\}$ is called the bond and serves as a numeraire; the bond may
here be risky or nonrisky. The other $d$ assets are called stocks and are described by the $d$-dimensional price process $\{S_t, 0 \leq t \leq T\}$. An investor is considered whose attitude towards risk is specified in terms of a utility. Here we restrict attention to strictly concave utility functions $U$ defined on the positive half-line with derivative $U'$. The utility being defined only for a nonnegative wealth, a dynamic portfolio is called admissible if its (self-financed) terminal wealth $V_T(x)$ depending on the initial wealth $x$ is nonnegative. It is convenient to specify a dynamic portfolio by a portfolio process $\pi$ (see Karatzas & Kou 1996) where the invested quantities are given in terms of percentages. Then a portfolio process $\pi$ can be used both by a rich and by a poor investor and one has the useful relation $V_T^\pi(x) = x \cdot V_T^\pi(1)$. The investor’s objective is to maximize the expected utility of terminal wealth and the maximum utility is given by

$$J(x) = \sup_\pi E[U(V_T^\pi(x))] = E[U(V_T^\pi(x))]$$

(1.1)

where $\pi^*$ is a solution to the optimization problem. Utility optimization is now a classical subject. However, here we are interested in optimal portfolios which can be chosen from the interior of the set of admissible portfolios. Hakansson (1971) has such a result under the assumption that $U(0+) = -\infty$ as in the case of the log utility. Here we present a result which makes use of $U'(0+) = \infty$ thus including the power functions $U_b(x) = x^{1-b}/(1-b)$ where $0 < b < 1$ is the index of (constant) proportional risk aversion. This utility is treated in Schäl (2000a) where the continuity and closedness assumptions (2.12 and 3.6) of the present paper are replaced by a quantitative assumption called uniform no-arbitrage assumption. Leland (1972), Bertsekas (1974), and Schäl (2000b) study models with utilities defined on the whole real line.

Optimal ‘interior’ portfolios can be used to define equivalent martingale measures. Since the martingale property is a local one both in time and in space, martingale measures can be constructed by local optimization problems. This was done by Rogers (1994) in order to prove the existence of a martingale measure under the no-arbitrage condition. The latter condition naturally plays an important role here, too. It was already used in a similar form by Hakansson (1971) under the name “no-easy-money-condition”. However, here we apply global optimization to obtain an equivalent martingale measure $Q^U_x$ by

$$dQ^U_x = \text{const} \cdot B_T \cdot U'(V_T^\pi(x))$$

where

$$\pi^* = \pi^*(x) \text{ is optimal for } x.$$  

(1.2)

For a proof of the existence of an optimal dynamic portfolio, the classical techniques known from discrete-time dynamic programming are used here. When applying the martingale method for portfolio optimization, the assumption $U'(0+) = \infty$ is also used in order to define the inverse of $U'$ (see Karatzas & Kou 1996, Davis 1997). However, only in the case of a finite underlying probability space, the condition $U'(0+) = \infty$ will turn out to be sufficient for the martingale property under $Q^U_x$. For more general spaces, a so-called Structure Assumption will be identified which implies the existence