A superlinearly convergent Newton-like algorithm for variational inequality problems with inequality constraints*

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Manuscript received: June 1999/Final version received: December 1999

Abstract. In this paper, a Newton-like method for variational inequality problems is considered. One feature of the algorithm is that only the solution of linear systems of equations is required at each iteration and that the strict complementarity assumption is never invoked. Another is that under mild assumptions, the sequence produced by the Newton-like method Q-superlinearly converges to the solution of the VIP. Furthermore, a simpler version of this method is studied and it is shown that it is also superlinearly convergent.

Key words: Newton-like method, variational inequality problems, superlinear convergence

1 Introduction

We consider the variational inequality problem, denoted by VIP(X,F), which is to find a vector $x^* \in X$ such that

$$F(x^*)^T(x^* - x) \geq 0, \quad \forall x \in X,$$

(1)

where $X$ is a nonempty closed convex subset of $R^n$ and $F(x)$ is a mapping from $R^n$ to $R^n$.

This problem has important applications, for example, in various equilibrium models arising in economics, operations research, transportation and regional science [1,2]. So much research has been done on this problem. A good survey of theory and algorithms for the variational inequality problem can be found in [2]. One popular approach is to reformulate the variational inequality problem as a nonlinear and usually nonsmooth equation. Pang

* This work is supported by the National Natural Science Foundation of China.
Xiao and Harker [11,12], for example, consider methods of this kind. Another interesting approach is based on optimization techniques. Two examples are the gap-function approach by Marcotte and Dussault [13] and the merit-function approach by Fukushima [6]. In particular, Fukushima’s method has motivated several authors to generalize his results, see, e.g., [3,4,9,14,15]. A third way to the solution of the variational inequality problem is continuation methods, Wang [16] describes embedding methods in his Ph.D. Thesis. Chen and Harker [8] present a continuation method for general monotone variational inequalities which depends on four perturbation parameters. Kanjowz and Jiang [18] recently present an interior-point-like continuation method for (strongly) monotone variational inequalities which depends on just one perturbation parameter.

Corresponding to the aforesaid approaches, there exist many methods (the Newton method, the projection method, the successive quadratic programming algorithm) for solving the variational inequality problem (1), but a very few papers mention the rate of convergence. Here we present a Newton-like method for solving VIP(X,F) with inequality constraints, i.e., where \( X = \{ x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, \ldots, m \} \), \( g_i : \mathbb{R}^n \rightarrow \mathbb{R} \) is a twice continuously differentiable convex function. One feature of this method is that only the solution of linear systems of equations is required at each iteration. Another is that the Q-superlinear convergence rate of this algorithm is proved without the strict complementarity assumption. Then a simpler version of this method is studied and it is shown that it is also superlinearly convergent.

This paper is organized as follows. In section 2 we precisely restate some basic definitions and the KKT conditions of VIP(X,F) and the perturbation set of the active constraint set. A Newton-like method for VIP(X,F) is proposed in section 3 and this algorithm is proved to be locally and superlinearly convergent. In section 4, a simpler version of this method is presented and shown to be superlinearly convergent.

2 Definitions and some basic results

In this paper we consider VIP(X,F) with inequality constraints, which is to find a vector \( x^* \in X \) such that

\[
F(x^*)^T(x^* - x) \geq 0, \quad \forall x \in X,
\]

where \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is continuously differentiable, \( X \) has a simple structure and can be described as follows:

\[
X = \{ x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, \ldots, m \},
\]

where \( g_i : \mathbb{R}^n \rightarrow \mathbb{R} (i = 1, \ldots, m) \) is a twice continuously differentiable convex function.

We first recall the definition of a (strongly) monotone function.

**Definition 2.1.** A function \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is called:

(a) monotone if

\[
(x - y)^T(F(x) - F(y)) \geq 0,
\]

for all \( x, y \in \mathbb{R}^n \).