On the existence and connectedness of solution sets of vector variational inequalities\textsuperscript{1}


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Abstract. In this paper, we consider a more general form of weak vector variational inequalities and prove some results on the existence of solutions of our new class of weak vector variational inequalities in the setting of Hausdorff topological vector topological space. We also discuss the connectedness of a solution set to some vector variational inequalities, and obtain a useful sufficient condition.

Key words: Vector Variational Inequality, Minty Lemma, Connectedness.

1 Introduction

The Vector Variational Inequality (for short, VVI) has been introduced in [8] in the setting of finite dimension Euclidean space. Because it has shown applications in different areas, including Optimization, Mathematical Programming, Operations Research and Economics, many authors have intensively studied it, for example [4, 5, 9, 10, 11], and their references therein.

During the last ten years, VVI and Weak VVI (for short WVVI) received significant attention, since they are closely related to Vector Optimization Problem (for short, VOP). Inspired and motivated by the applications of WVVI, in the first part of this paper, we consider a more general form of the WVVI. By using the famous KKM theorem and a generalized linearization lemma which is a generalization of Minty lemma, we prove some results on the existence of solutions of our new class of WVVI in the setting of topological vector space. Moreover, in the second part of this paper, we put our emphasis on the connectedness of solution sets for some VVIs. And we obtain the sufficient conditions of connectedness for these VVIs by defining a kind of scalar variational inequality.

Let $Y$ be a Hausdorff topological vector space (over reals).

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Definition 1.1. The Hausdorff topological vector space $Y$ is said to be an ordered Hausdorff topological vector space denoted by $(Y, P)$ if ordering relations are defined in $Y$ by a closed convex cone $P$ of $Y$ as follows:

$$\forall x, y \in Y, \quad y \leq x \iff x - y \in P$$

$$\forall x, y \in Y, \quad y \leq x \iff x - y \in P \setminus \{0\}$$

$$\forall x, y \in Y, \quad y \not\leq x \iff x - y \not\in P \setminus \{0\}$$

If the interior $int P$ of $P$ is nonempty, then the weak ordering relations in $Y$ are also defined as follows:

$$\forall x, y \in Y, \quad y < x \iff x - y \in int P$$

$$\forall x, y \in Y, \quad y \not< x \iff x - y \not\in int P$$

Let $X$ be another Hausdorff topological vector space (over reals). Let $L(X, Y)$ be the set of all linear continuous functions from $X$ to $Y$. For $l \in L(X, Y)$, the value of the linear function $l$ at $x$ is denoted by $\langle l, x \rangle$. Let $Y^*$ be the dual space of $Y$, and $X^*$ be the dual space of $X$. Let $P^* = \{f \in Y^*: f(x) \geq 0, \forall x \in P\}$ be the dual cone of $P$.

Throughout this paper, we assume that $(Y, P)$ is an ordered Hausdorff topological vector space, $int P \neq \emptyset$ and $X^*, Y^*$ are equipped with weak* topology, otherwise specified.

Firstly, let us recall the Knaster-Kuratowski-Mazurkiewicz theorem.

Lemma 1.1 (KKM Theorem). Let $E$ be a subset of the topological vector space of $X$ and $F : X \to 2^X$. For each $x \in E$, let $F(x)$ be closed, and $F(x)$ be compact for at least one $x \in E$. If the convex hull of every finite subset $\{x_1, \ldots, x_n\}$ of $E$ is contained in the corresponding union $\bigcup_{i=1}^n F(x_i)$, then $\bigcap_{x \in E} F(x) \neq \emptyset$.

If $F$ is called a KKM mapping if for every finite subset $\{x_1, \ldots, x_n\}$ of $E$, we have

$$CO\{x_1, \ldots, x_n\} \subseteq \bigcup_{i=1}^n F(x_i)$$

Now, we introduce a new vector variational inequality denoted by GWVVI: for each $z \in K, \lambda \in (0, 1]$, find $x_0 \in K$, such that

$$\langle T(\lambda x_0 + (1 - \lambda)z) - x_0 \rangle \neq \text{int } P, \quad \forall y \in K,$$

where $T : X \to L(X, Y)$ is a function and $K$ is a subset of $X$.

Obviously, if $\lambda = 1$, Problem (GWVVI) is the usual (VVVI), i.e., finding $x_0 \in K$, such that

$$\langle Tx_0 - x_0 \rangle \neq \text{int } P, \quad \forall y \in K.$$ 

Further, for any $z \in K, \lambda \in (0, 1]$, let $x_0 = x(\lambda, z)$ be denoted a solution of problem (GWVVI). Clearly, $x(1, z)$ is the solution of problem (VVVI).

Naturally, we consider such a problem: whether $x(\lambda, z) \to x(1, z)$, when $\lambda \to 1^-$? This problem will be discussed in another article.